Math 336 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover through section III.5 in Gamelin. You may assume that a holomorphic function is C^2 on its domain of definition. $\mathbb{D} = \{z : |z| < 1\}$ denotes the open unit disk. $\mathbb{H} = \{z = x + iy : y > 0\}$ denotes the open upper half plane.

1. Suppose u is harmonic on $\mathbb{D}$ and continuous on the closure of $\mathbb{D}$. If $u(z) = 0$ when $z \in \partial\mathbb{D}$, what can you say about $\mathbb{D}$. Suppose v is harmonic on $\mathbb{H}$ and continuous on the closure of $\mathbb{H}$. If $v(z) = 0$ when $z \in \partial\mathbb{H}$, what can you say about v?

2. Let
$$f(z) = \begin{cases} 1 \text{ if } xy = 0, \\ 0 \text{ if } xy \neq 0. \end{cases}$$
Find all points at which f is complex differentiable. Be careful.

3. Find all zeros of the complex logarithm function $\log(z)$. (Find all points z at which $\log(z) = 0$.)

4. Show that the linear fractional transformation $w = Tz$ that maps z_1, z_2, z_3 to w_1, w_2, w_2 is determined by the equality of cross ratios
$$(w, w_1, w_2.w_3) = (z, z_1, z_2, z_3)$$
.

5. Let $T(z) = \frac{1}{2\cos(\frac{\pi}{n}) - z}$. Prove that $T^n(z) = z$. Does this contradict anything? T^n means T composed with itself n times. (The previous problem might help in solving this problem.)

6. Suppose $w \neq 1, w \in \mathbb{C}$ is an n^{th} root of unity, $w^n = 1$. Prove that
$$1 + 2w + 3w^2 + \cdots + nw^{n-1} = \frac{n}{w-1}.$$

7. Suppose $Re(z_j) \geq 0$ for $j \geq 1$ and suppose the series
$$z_1 + z_2 + \cdots + z_n + \ldots$$
$$z_1^2 + z_2^2 + \cdots + z_n^2 + \ldots,$$
both converge. Prove that
$$|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 + \ldots,$$
converges.

8. Let $f(z) = x^2 - y^2 + i\log(x^2 + y^2)$. Find the points at which f is complex differentiable. Find the points at which $g(z) = x - iy$ is complex analytic.

9. Let $f(z) = u(z)+iv(z), u = Re(f(z)), v = Imf((z))$ be analytic on an open connected set Ω. Suppose there are real numbers a, b, c with $a^2 + b^2 \neq 0$ and $au(z) + bv(z) = c$ for all $z \in \Omega$. Prove that f is constant.

10. Suppose that v is the harmonic conjugate of u and u is the harmonic conjugate of v. Show that u and v must be constant.

11. Let u be harmonic on W (assume u is twice continuously differentiable). Prove that $f(z) = u_x(z) - iu_y(z)$ is analytic.

12. Let a be a complex number and suppose $|a| < 1$. Let $f(z) = \dfrac{z-a}{1-\bar{a}z}$. Prove the following statements.

    (a) $|f(z)| < 1$, if $|z| < 1$.
    (b) $|f(z)| = 1$, if $|z| = 1$.

13. Suppose $f = u + iv$ is analytic on $\{Re(z) > 0\}$ and $u_x + v_y = 0$. Prove that there is a real number c and complex number d so that
$$f(z) = icz + d.$$

14. Let $f(z) = e^{-z^{-4}}$ if $z \neq 0$, $f(0) = 0$. Prove that f is analytic at z if $z \neq 0$ and that the Cauchy-Riemann equations are satisfied at 0. Is f analytic at 0?

15. Let $z_j = e^{\frac{2\pi ij}{n}}$ denote the n roots of unity. Let $c_j = |1 - z_j|$ be the $n-1$ chord lengths from 1 to the points $z_j, j = 1, \ldots .n-1$. Prove that the product $c_1 \cdot c_2 \cdots c_{n-1} = n$. *Hint*: Consider $z^n - 1$.

16. Find a sequence of complex numbers z_n such that $\sum_{n=1}^{\infty} z_n^k$ converges for every $k = 1, 2 \ldots$ but $\sum_{n=1}^{\infty} |z_n|^k$ diverges for every $k = 1, 2 \ldots$. Hint: Try $z_n = \frac{e^{2\pi ins}}{\log(n+1)}$ for an appropriate real number s.

17. Suppose f is analytic on a connected open set. Assume $f^2 = \overline{f}$. Prove that f is constant. What are the possible values of the constant?

18. You will need to know the definitions of the following terms and statements of the following theorems.

    (a) Modulus (absolute value) and argument of a complex number
    (b) Complex derivative
    (c) Complex analytic function
    (d) Cauchy-Riemann equations

(e) Harmonic functions and harmonic conjugate
(f) Complex exponential function and trigonometric functions
(g) Complex logarithm and powers
(h) Linear fractional transformations
(i) Mean value property
(j) Maximum principle

13. There may be homework problems or example problems from the text on the midterm.