Schwarzschild Solution to Einstein's General Relativity

Carson Blinn

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Abstract

This paper is intended as a very brief review of General Relativity for those who do not want to skimp on the details of the mathematics behind how the theory works. This paper mainly uses [2], [3], [4], and [6] as a basis, and in addition contains short references to more in-depth references such as [1], [5], [7], and [8] when more depth was needed. With an introduction to manifolds and notation, special relativity can be constructed which are the relativistic equations of flat space-time. After flat space-time, the Lagrangian and calculus of variations will be introduced to construct the Einstein-Hilbert action to derive the Einstein field equations. With the field equations at hand the Schwarzschild equation will fall out with a few assumptions.

1 Introduction

Einstein's General Relativity is a powerful physical theory that describes interactions in the universe in much greater accuracy than the previous Newtonian theory of gravitation. Light is established as the invariant "speed limit" of causality, described by Lorentz invariant transformations, which are the baseline assumption holding up General Relativity. In this paper, tensor notations, Lorentz contractions, and Minkowski space will be introduced in order to lay a foundation for understanding the Einstein Field Equations taken directly from Einstein's first paper [3], and these tools will be utilized to derive the Einstein equations and the Schwarzschild solution to the equations and understand their implications on physical phenomena.

1.1 Tensor Notations

An arbitrary tensor $A_{\mu\nu}$ that acts on 4-vectors (which is what is used for space-time vectors) is given by

$$A_{\mu\nu} = \begin{pmatrix} A_{00} & A_{10} & A_{20} & A_{30} \\ A_{01} & A_{11} & A_{21} & A_{31} \\ A_{02} & A_{12} & A_{22} & A_{32} \\ A_{03} & A_{13} & A_{23} & A_{33} \end{pmatrix}$$

Which is the standard size used in General Relativity, where the first columns $(0, 1, 2, 3)$ correspond to (t, x, y, z).

A vector V^μ is called a covariant vector, it is analogous to a normal vector. A vector with the index is the subscript V_μ is called contravariant and instead acts like a differential form. Multiplying these two types of vectors gives a form of dot product.

When two vectors or tensors with upper and lower indices are multiplied, a sum sign is implied:

$$A_{\theta\theta} = \sum_{n=0}^{3} A_{nn} = \mathrm{Tr}(A)$$

$$df = \frac{\partial f}{\partial x_\alpha} dx^\alpha = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

The metric tensor giving the Lorentz transformation metric is $g_{\mu\nu}$. This tensor is symmetric;

$$g_{\mu\nu} = g_{\nu\mu}$$

And has the form

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Such that $ds^2 = -\,dt^2 + \,dx^2 + \,dy^2 + \,dz^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$.
Tensor transformation laws to primed coordinates take the form

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$$

The inverse of $g_{\mu\nu}$ is signified by $g^{\mu\nu}$, such that

$$g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$$

(The metric tensor will be expanded upon in the derivation of the Einstein Field Equations [Section 3]) A more in depth discussion of this topic can be found in [5].

1.2 Manifolds

Manifolds are a necessary topic of General Relativity as they mathematically define the curvature and surface characteristics of a space that the mathematician/physicist is working in. General Relativity asserts that all space-time takes place on a curved manifold, where particles move along geodesics (shortest path between two points) in the curvature of space-time.

The following introduction is adapted from [2].

A manifold is created by local mappings from a coordinate system (open set $U \subset M$) into the vector space in question. A collection of local mappings (usually denoted $\varphi_i : U_i \rightarrow \mathbb{R}^n$) that completely covers a manifold with open sets is called an atlas.
Locally, every manifold (the ones we will consider in General Relativity) looks flat, so analysis can be done.

A consequence of the definition of a tensor is that the partial derivative of a tensor does not output a tensor. Therefore, a new derivative must be defined so that tensors moving along geodesics can have workable derivative-like operators; this is called the covariant derivative. The covariant derivative on a contravariant vector is defined as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

or

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda$$

for a covariant vector; for tensors a matching connection coefficient is added for each index.
$\partial_\mu V^\nu$ is the partial derivative and $\Gamma^\nu_{\mu\lambda} V^\lambda$ is the correction to keep the derivative in tensor form. $\Gamma^\nu_{\mu\lambda}$ is the connection coefficient, which is given by the metric. Connection coefficients are antisymmetric in their lower indices.

The connection derived from this metric is called the Levi-Civita connection, or the Riemannian connection. It is simple to prove existence and uniqueness of the connection coefficient:

Proof. First expand the equation for metric compatibility ($\nabla_\rho g_{\mu\nu} = 0$) in three different permutations of the indices:

$$\begin{aligned}
\nabla_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0 \\
\nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \\
\nabla_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0
\end{aligned}$$

Subtracting the last two equalities from the first and using the symmetric nature of the connection coefficients results in

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0$$

Solving for the connection by multiplying by $g^{\sigma\rho}$:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda} \left(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda} \right)$$

where $g_{\mu\nu,\lambda} = \partial_{\lambda} g_{\mu\nu}$.

$\square$

This means that each connection symbol is unique and can be calculated from the metric.

2 Lorentz Transformations

Moving from introduction to analysis of the physical aspects of the theory, the Lorentz Transformations take into account the effects of general relativity on a flat space-time, and make up the basis of Einstein's special relativity. Einstein initially formulated these equations, and then took many years to develop the analog to these transformations on a curved space-time.

2.1 Characteristic equations

Before Einstein, Newtonian relativity was taken as fact, which accounted for the 3 spatial dimensions:

$$ds^2 = dx^2 + dy^2 + dz^2$$

which assumes no change under rotation of coordinates and preservation of norm for all coordinate systems, and relative velocities are simply added and subtracted between frames of reference.

$$\begin{aligned} V^{x'} &= V^{x} - v \\ V^{y'} &= V^{y} \\ V^{z'} &= V^{z} \end{aligned}$$

For a change in velocity v between frames, using Newtonian addition of velocities.

This notion was changed after the Michelson-Morley experiment, which used two light beams in an interferometer; one rotated at high speeds and was compared to a reference beam in order to detect a change in frequency of the light according to the thought at the time that light traveled through a medium. Since no difference was detected, the Newtonian addition of velocities was inaccurate in the case of light.

Einstein suggested that light's speed was an invariant, and that the Galilean coordinate transformations were flawed at near light velocities; replacing them with the Lorentz Transformations. These transformations keep the space-time interval constant, defined as

$$s^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{1}$$

Which gives the definition of distance in Minkowski (Lorentz invariant space)

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{2}$$

Einstein's simple proof of the Lorentz transformations [4]:

Proof. Assume without loss of generality that the particle in question is moving along the x axis.
Take a light signal traveling along the x axis:

$$x = ct \Rightarrow x - ct = 0$$

Since the speed of light is invariant;

$$x' = ct' \Rightarrow x' - ct' = 0$$

Events in any frame must fulfill either equation, so the equations must fulfill the relation

$$(x' - ct') = \lambda(x - ct)$$

For light running in the negative direction on the x axis, the relations becomes

$$(x' + ct') = \mu(x + ct)$$

Redefining the constants as

$$\gamma = \frac{\lambda + \mu}{2}, \qquad b = \frac{\gamma - \mu}{2}$$

$$x' = \gamma x - bct, \qquad ct' = \gamma ct - bx$$

Taking x' to be the system at the origin, the relative velocity can be found by

$$0 = ax - bct \Rightarrow \frac{x}{t} = \frac{bc}{a} = v$$

Rearranging the equations gives the Lorentz transformations

$$
\begin{aligned}
x' &= \gamma(x - vt) \\
t' &= \gamma\left(t - \frac{v}{c^2}x\right) \\
y' &= y \\
z' &= z
\end{aligned}
$$

Gamma is given by the fact that at $t = 0$:

$$x' = \gamma x$$

While at $t' = 0$:

$$0 = \gamma\left(t - \frac{v}{c^2}x\right), \qquad x' = \gamma(x - vt) \Rightarrow t = \frac{x' - \gamma x}{-v\gamma}$$

Subbing in for t:

$$0 = \gamma\left(\frac{x' - \gamma x}{-v\gamma} - \frac{v}{c^2}x\right) \Rightarrow x' = \gamma\left(1 - \frac{v^2}{c^2}\right)x$$

Since $\frac{x'}{x} = \frac{x}{x'}$ by the relativistic assumption,

$$\frac{1}{\gamma} = \gamma\left(1 - \frac{v^2}{c^2}\right) \Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$\square$

These transformations show that as the velocity of a particle approaches the speed of light, the relative length of the particle will be contracted in a stationary observer's reference frame, while the stationary observer will observe more time passing in the particle's frame of reference.

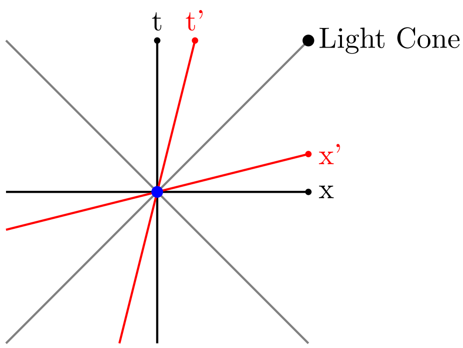
2.2 Minkowski Space

Minkowski Space is four dimensional real space coordinatized by (t, x, y, z), where the inner product is given by the space-time interval given above: $s^2 = -c^2t^2 + x^2 + y^2 + x^2$. This encodes the Lorentz transformations along just the x axis to 4 dimensional flat space, which is the space we live in. This space is simply a local representation of the space that our universe is set in, such that Einstein's postulate on the invariance of the speed of causality is invariant.

Minkowski Space represents space-time with zero curvature: it only represents special relativity. However, on every manifold of General Relativity local regions look like Minkowski Space.

A Minkowski Diagram displays the light cone of an observer at 45 degrees–it shows what evens could be causally related to the observer (anything outside the light cone could in no way communicate with the observer). The diagrams can be transformed to new coordinates with Lorentz boosts, but the speed of light with always point at 45 degrees in a light cone from the observer.

Sample Minkowski Diagram:



The central dot is the observer, while the t' and x' lines show the transformation of coordinates under a Lorentz transform. The speed of light $c = 1$ for simplicity. Points correspond to events, and events above the light cone have a time like space-time interval ($s^2 < 0$), while events below the light cone have a space like space-time interval ($s^2 > 0$). Points along the light cone have light-like or null space-time interval ($s^2 = 0$).

3 Derivation of the Einstein Field Equations

3.1 Calculus of Variations

This introduction is paraphrased from [1].
Moving from flat spaces to curvature, a method for finding the shortest path along a manifold becomes a non-trivial pursuit. A new method must be derived to find the shortest path, and that method is Calculus of variations. Calculus of variations was developed with the purpose of finding a method

to solve for the shortest path between two points on a curved surface. It is equivalent to finding stationary points of the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\,dx \qquad \left(y' = \frac{dy}{dx}\right)$$

Assuming that all curves are of the form

$$Y(x) = y(x) + \epsilon\eta(x),$$

where $y(x)$ is the geodesic and $\eta(x)$ are small variations from the minimum value. $\eta(x) = 0$ at the endpoints of each curve.

The geodesic is the solution where I is minimized when $\epsilon = 0$, or equivalently solving

$$\frac{dI}{d\epsilon} = 0, \text{ and } \epsilon = 0$$

simultaneously.

Solving in generality the integral we can arrive at Euler's equation:

Proof.

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, Y, Y')\,dx$$

Differentiating under the integral (which is valid since F is sufficiently smooth in regards to x, y, and y'.

$$\frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y}\frac{dY}{d\epsilon} + \frac{\partial F}{\partial Y'}\frac{dY'}{d\epsilon}\right] dx$$

$$\left(\frac{dY}{d\epsilon} = \eta(x) \text{ and } \frac{dY'}{d\epsilon} = \eta'(x) \text{ from } Y(x) = y(x) + \epsilon\eta(x)\right)$$

$$\frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y}\eta(x) + \frac{\partial F}{\partial Y'}\eta'(x)\right] dx$$

Solving for $\epsilon = 0$ gives $Y(x) = y(x)$ and $\dfrac{\partial I}{\partial \epsilon} = 0$:

$$\frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right] dx$$

Assuming y'' is continuous (which is usually the case in physical systems) the second term can be integrated by parts:

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'(x)\, dx = \frac{\partial F}{\partial y'} \eta(x) \bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \eta(x)\, dx$$

Since $\eta(x)$ is defined as zero at the endpoints, the first integrated term disappears. This leaves

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \eta(x)\, dx = 0$$

Since $\eta(x)$ is arbitrary, the other term must be identically equal to zero:

$$\frac{d}{dx} \frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

which is Euler's equation.

□

The notation

$$\delta I = \frac{dI}{d\epsilon} d\epsilon$$

is the classical notation that this paper will use.

3.2 Einstein-Hilbert Action

Sources used were [8], and [2]. David Hilbert realized that he could use classical mechanics of geodesic calculation to create Einstein's field equations on a mathematical basis. The Lagrangian (denoted $\mathcal{L}$) is a function that encodes all the dynamics of a system, and when integrated gives an action, denoted as $\mathcal{S}$.

$$\mathcal{S} = \int \mathcal{L}\, d^4x$$

The action is a functional that takes the trajectory (also called path or history) of an object as an argument and returns a scalar. The principle of least action states that objects in motion follow the path which minimizes the action ($\delta S = 0$, this uses calculus of variations).

Hilbert observed that the action of space-time must take the form

$$\mathcal{S} = \int \left[\frac{1}{2\kappa} R + \mathcal{L}_M \right] \sqrt{-g}\, d^4x$$

where R is the Ricci scalar and $\mathcal{L}_M$ is the contributions from mass densities to the Lagrangian. g is the determinant of the metric.
Since objects travel through space-time along geodesics, the least action principle should describe motion in space-time:

$$\delta\mathcal{S} = \int \left[\frac{1}{2\kappa}\frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\, d^4x = 0$$

$$= \int \left[\frac{1}{2\kappa}\left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right) + \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\, d^4x$$

Since the equation should be true for all variations $\delta g^{\mu\nu}$, this implies that

$$\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -2\kappa\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}$$

The right hand side of the above equation is defined as proportional to the stress energy tensor:

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M$$

3.3 Calculation of the Variation of the Ricci Tenosr and Scalar

The Riemann curvature tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

is needed to find the variation of the Ricci tensor and scalar. First the variation $\delta R^\rho{}_{\sigma\mu\nu}$ will be calculated and then contracted into the needed objects.

$$\delta R^\rho{}_{\sigma\mu\nu} = \partial_\mu\delta\Gamma^\rho_{\nu\sigma} - \partial_\nu\delta\Gamma^\rho_{\mu\sigma} + \delta\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\sigma} - \delta\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\mu\sigma}$$

using the product rule and linearity of derivatives.
$\delta\Gamma^\rho_{\nu\mu}$ is the difference of two Christoffel symbols and is therefore a tensor, so it has a covariant derivative:

$$\nabla_\lambda\left(\delta\Gamma^\rho_{\nu\mu}\right) = \partial_\lambda\delta\Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\sigma\lambda}\delta\Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\nu\lambda}\delta\Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\mu\lambda}\delta\Gamma^\rho_{\nu\sigma}$$

In terms of the above equation for covariant derivatives, the variation of the Riemann curvature tensor is

$$\delta R^\rho{}_{\sigma\mu\nu} = \nabla_\mu\left(\delta\Gamma^\rho_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^\rho_{\mu\sigma}\right)$$

Contracting gives

$$\delta R_{\mu\nu} \equiv \delta R^{\rho}{}_{\mu\rho\nu} = \nabla_\rho \left(\delta\Gamma^{\rho}_{\nu\mu}\right) - \nabla_\nu \left(\delta\Gamma^{\rho}_{\rho\mu}\right)$$

For the Ricci Scalar, $R = g^{\mu\nu} R_{\mu\nu}$:

$$\begin{aligned}\delta R &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \\ &= R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\sigma \left(g^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^{\rho}_{\rho\mu}\right)\end{aligned}$$

which is true from the equation of the variation of the Ricci scalar and the fact that $\nabla_\sigma g^{\mu\nu} = 0$. $\nabla_\sigma \left(g^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^{\rho}_{\rho\mu}\right)$ disappears for unbounded solutions, so

$$\therefore \frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}$$

Jacobi's formula describs the method for differentiating a determinant:

$$\delta g = \delta \det(g_{\mu\nu}) = g g^{\mu\nu} \delta g_{\mu\nu}$$

Using this on the $\sqrt{-g}$ component:

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}(g^{\mu\nu}\delta g_{\mu\nu}) = -\frac{1}{2}\sqrt{-g}(g_{\mu\nu}\delta g^{\mu\nu})$$

(the last equality is due to the fact that $g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}$)

$$\therefore \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu}$$

3.4 The Einstein Equations

Simply plugging in the expressions derived in the the last section to

$$\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -2\kappa \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}$$

gives the equations that we are looking for:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Where $\kappa = \dfrac{8\pi G}{c^4}$ so that the equations match the Newtonian limit.

4 Derivation of Schwarzschild Metric

4.1 Assumptions

1. The Schwarzschild metric assumes that the system is spherically symmetric; it uses spherical coordinates along the metric to achieve this symmetry (it can be seen with the r^2 and $r^2 \sin^2\theta$ terms of the metric).
2. The solution assumes vacuum conditions ($T_{\mu\nu} = 0$), so that solutions only have to solve $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$.
3. The solution assumes that the system is static and time invariant, proven with Birkhoff's theorem, which is too complicated of an argument for this paper which I leave the details to [7]. This allows $g_{00} = U(r,t)$ and $g_{11} = -V(r,t)$ to be limited to functions of only r: $g_{00} = U(r)$, and $g_{11} = -V(r)$.

4.2 Deriving the Christoffel Symbols

This follows the method presented in [6].
Using the spherical parameterization of the metric:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \, dt^2 - \, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\psi^2$$

Where

$$g^\mu{}_\mu = g^{\mu\nu} g_{\mu\nu} = 4$$

Generalizing this with functions on each of the infinitesimals

$$ds^2 = U\, dt^2 - V\, dr^2 - W r^2\, d\theta^2 - X r^2 \sin^2\theta\, d\psi^2$$

However, since it was assumed that the equations are spherically symmetric, $W = X = 1$. Since the solution is for a static field, the functions have no dependence on time, and since the only mass is located inside a point mass, the stress energy tensor ($T_{\mu\nu}$) will vanish.

$$\therefore\ ds^2 = U(r)\, dt^2 - V(r)\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\psi^2$$

However, the problem is reduced in complexity to solutions of

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

because $T_{\mu\nu} = 0$ from the assumptions.
By the above reasoning, the metric tensor $g_{\mu\nu}$ is defined as

$$g_{00} = U, \qquad g_{11} = -V, \qquad g_{22} = -r^2, \qquad g_{33} = -r^2 \sin^2\theta$$

and its inverse $g^{\mu\nu}$:

$$g^{00} = \frac{1}{U}, \qquad g_{11} = \frac{-1}{V}, \qquad g^{22} = \frac{-1}{r^2}, \qquad g^{33} = \frac{-1}{r^2 \sin^2\theta}$$

To find the Ricci Scalar and Tensor, the Riemann curvature tensor must be calculated. It is given by

$$R^{\beta}{}_{\nu\rho\sigma} = \Gamma^{\beta}_{\nu\sigma,\rho} - \Gamma^{\beta}_{\nu\rho,\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\beta}_{\alpha\sigma}$$

where variables after the comma signify derivatives $\Gamma^{\beta}_{\nu\sigma,\rho} = \frac{\partial}{\partial x^{\rho}}\Gamma^{\beta}_{\nu\sigma}$, and

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}\left(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}\right)$$

as defined above.
To simplify the calculation of the Christoffel Symbols;

1. Any derivatives with respect to t are zero, as the solution is static and does not depend on time.

2. $g_{\mu\nu}$ and $g^{\mu\nu}$ both equal zero when $\mu \neq \nu$ (the metric is symmetric).

3. $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$ (the Christoffel Symbols are symmetric in their lower indices)

(indices μ and ν run from $0-3$, while i and j run from $1-3$ in the spatial dimensions). $\Gamma^{0}_{\mu\nu}$

$$\Gamma^{0}_{00} = \frac{1}{2}g^{00}\left(g_{00,0} + g_{00,0} - g_{00,0}\right) = 0$$
$$\Gamma^{0}_{0i} = \frac{1}{2}g^{00}\left(g_{00,i} + g_{0i,0} - g_{0i,0}\right) = \frac{1}{2}g^{00}\partial_1 g_{00} = \frac{1}{2}\frac{1}{U}\partial_r U$$
$$\Gamma^{0}_{ij} = \frac{1}{2}g^{00}\left(g_{00,0} + g_{00,0} - g_{00,0}\right) = 0$$

$\Gamma^1_{\mu\nu}$

$$\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2}g^{11}\left(g_{10,0}+g_{10,0}-g_{00,1}\right) = -\frac{1}{2}g^{11}\partial_1 g_{00} = \frac{1}{2}\frac{1}{V}\partial_r U \\
\Gamma^1_{0i} &= \frac{1}{2}g^{11}\left(g_{10,i}+g_{1i,0}-g_{0i,1}\right) = 0 \\
\Gamma^0_{ij\neq i} &= \frac{1}{2}g^{11}\left(g_{00,0}+g_{00,0}-g_{00,0}\right) = 0 \\
\Gamma^1_{11} &= \frac{1}{2}g^{11}\left(g_{11,1}+g_{11,1}-g_{11,1}\right) = \frac{1}{2}g^{11}\partial_1 g_{11} = \frac{1}{2}\frac{1}{V}\partial_r V \\
\Gamma^1_{22} &= \frac{1}{2}g^{11}\left(g_{12,2}+g_{12,2}-g_{22,1}\right) = -\frac{1}{2}g^{11}\partial_1 g_{22} = -\frac{r}{V} \\
\Gamma^1_{33} &= \frac{1}{2}g^{11}\left(g_{13,3}+g_{13,3}-g_{33,1}\right) = -\frac{1}{2}g^{11}\partial_1 g_{33} = -\frac{r}{V}\sin^2\theta
\end{aligned}$$

$\Gamma^2_{\mu\nu}$

$$\begin{aligned}
\Gamma^2_{00} &= \frac{1}{2}g^{22}\left(g_{20,0}+g_{20,0}-g_{00,2}\right) = 0 \\
\Gamma^2_{0i} &= \frac{1}{2}g^{22}\left(g_{20,i}+g_{2i,0}-g_{0i,2}\right) = 0 \\
\Gamma^2_{ii} &= \frac{1}{2}g^{22}\left(g_{2i,i}+g_{2i,i}-g_{ii,2}\right) = -\frac{1}{2}g^{22}\partial_2 g_{33} = -\cos\theta\sin\theta \\
\Gamma^2_{12} &= \frac{1}{2}g^{22}\left(g_{21,2}+g_{22,1}-g_{12,2}\right) = \frac{1}{2}g^{22}\partial_1 g_{22} = \frac{1}{r} \\
\Gamma^2_{13} &= \frac{1}{2}g^{22}\left(g_{21,3}+g_{23,1}-g_{13,2}\right) = 0 \\
\Gamma^2_{23} &= \frac{1}{2}g^{22}\left(g_{22,3}+g_{23,2}-g_{23,2}\right) = 0
\end{aligned}$$

$\Gamma^3_{\mu\nu}$

$$\Gamma^3_{00} = \frac{1}{2}g^{33}\left(g_{30,0} + g_{30,0} - g_{00,3}\right) = 0$$
$$\Gamma^3_{0i} = \frac{1}{2}g^{33}\left(g_{30,i} + g_{3i,0} - g_{0i,3}\right) = 0$$
$$\Gamma^3_{ii} = \frac{1}{2}g^{33}\left(g_{3i,i} + g_{3i,i} - g_{ii,3}\right) = 0$$
$$\Gamma^3_{ij\neq i} = \frac{1}{2}g^{33}\left(g_{3i,j} + g_{3j,i} - g_{ij,3}\right) = \frac{1}{2}g^{33}\left(g_{3i,j} + g_{3j,i}\right) :$$
$$\Gamma^3_{13} = \frac{1}{2}g^{33}\left(g_{31,3} + g_{33,1}\right) = \frac{1}{2}g^{33}\partial_1 g_{33} = \frac{1}{r}$$
$$\Gamma^3_{23} = \frac{1}{2}g^{33}\left(g_{32,3} + g_{33,2}\right) = \frac{1}{2}g^{33}\partial_2 g_{33} = \frac{cos\theta}{\sin\theta}$$

All the non-vanishing terms: (primes denote ∂_r)

$$\Gamma^0_{01} = \Gamma^0 10 = \frac{U'}{2U}$$
$$\Gamma^1_{00} = \frac{U'}{2V}$$
$$\Gamma^1_{11} = \frac{V'}{2V}$$
$$\Gamma^1_{22} = -\frac{r}{V}$$
$$\Gamma^1_{33} = -\frac{r}{V}\sin^2\theta$$
$$\Gamma^2_{12} = \Gamma^0 21 = \frac{1}{r}$$
$$\Gamma^2_{33} = -\cos\theta\sin\theta$$
$$\Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r}$$
$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta}$$

4.3 The Ricci Tensor

The Ricci Tensor is a contraction of the Riemann curvature tensor;

$$R_{\mu\nu} = R^{\beta}{}_{\mu\nu\beta} = \Gamma^{\beta}_{\mu\beta,\nu} - \Gamma^{\beta}_{\mu\nu,\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}$$

$$
\begin{aligned}
R_{\mu\nu} =& \Gamma^0_{\mu 0,\nu} - \Gamma^0_{\mu\nu,0} + \Gamma^\alpha_{\mu 0}\Gamma^0_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^0_{\alpha 0} \\
&+ \Gamma^1_{\mu 1,\nu} - \Gamma^1_{\mu\nu,1} + \Gamma^\alpha_{\mu 1}\Gamma^1_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^1_{\alpha 1} \\
&+ \Gamma^2_{\mu 2,\nu} - \Gamma^2_{\mu\nu,2} + \Gamma^\alpha_{\mu 2}\Gamma^2_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^2_{\alpha 2} \\
&+ \Gamma^3_{\mu 3,\nu} - \Gamma^3_{\mu\nu,3} + \Gamma^\alpha_{\mu 3}\Gamma^3_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^3_{\alpha 3}
\end{aligned}
$$

$R_{\mu\nu}, \mu \neq \nu$

$$
\begin{aligned}
R_{0i} =& \Gamma^0_{00,i} - \Gamma^0_{0i,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^0_{\alpha 0} \rightarrow 0 - 0 + \Gamma^\alpha_{01}\Gamma^1_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^1_{\alpha 1} \\
&+ \Gamma^1_{01,i} - \Gamma^1_{0i,1} + \Gamma^\alpha_{01}\Gamma^1_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^1_{\alpha 1} \rightarrow 0 - 0 + 0 - 0 \\
&+ \Gamma^2_{02,i} - \Gamma^2_{0i,2} + \Gamma^\alpha_{02}\Gamma^2_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^2_{\alpha 2} \rightarrow 0 - 0 + 0 - 0 \\
&+ \Gamma^3_{03,i} - \Gamma^3_{0i,3} + \Gamma^\alpha_{03}\Gamma^3_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^3_{\alpha 3} \rightarrow 0 - 0 + 0 - 0
\end{aligned}
$$

Since either $\alpha = 0$, which makes the terms vanish because $\Gamma^0_{00} = 0$, or $\alpha = j$ where the second Christoffel symbol is equal to zero. Therefore, $R_{0i} = 0$.

$$
\begin{aligned}
R_{ij\neq i} =& \Gamma^0_{i0,j} - \Gamma^0_{ij,0} + \Gamma^\alpha_{i0}\Gamma^0_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^0_{\alpha 0} \rightarrow 0 - 0 + \Gamma^\alpha_{i0}\Gamma^1_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^0_{\alpha 0} \\
&+ \Gamma^1_{i1,j} - \Gamma^1_{ij,1} + \Gamma^\alpha_{i1}\Gamma^1_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^1_{\alpha 1} \rightarrow 0 - 0 + \Gamma^\alpha_{i1}\Gamma^1_{\alpha j} - 0 \\
&+ \Gamma^2_{i2,j} - \Gamma^2_{ij,2} + \Gamma^\alpha_{i2}\Gamma^2_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^2_{\alpha 2} \rightarrow 0 - 0 + \Gamma^\alpha_{i2}\Gamma^2_{\alpha j} - 0 \\
&+ \Gamma^3_{i3,j} - \Gamma^3_{ij,3} + \Gamma^\alpha_{i3}\Gamma^3_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^3_{\alpha 3} \rightarrow \Gamma^3_{i3,j} - 0 + \Gamma^\alpha_{i3}\Gamma^3_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^3_{\alpha 3}
\end{aligned}
$$

Looking at the rest of the terms consists of casework to make each go to zero.
$\Gamma^\alpha_{i0}\Gamma^0_{\alpha j}$: If α or $j = 2, 3$ the second Christoffel symbol goes to zero. The other options are $\alpha = 0$, where $i = 1$ for it to not vanish. However the condition that $i \neq j$ forces $j = 2, 3$ and the symbol goes to zero. For $\alpha = 1$, i would have to be zero, however i is only indexed from $1 - 3$, so the term as a whole is zero.
$-\Gamma^\alpha_{ij}\Gamma^0_{\alpha 0}$: The only nonvanishing term is if $\alpha = 1$ for the second symbol, however $\Gamma^1_{ij} = 0 \Rightarrow -\Gamma^\alpha_{ij}\Gamma^0_{\alpha 0} = 0$
$\Gamma^\alpha_{i1}\Gamma^1_{\alpha j}$: For terms to not immediately be zero $\alpha = j$, this leaves $\Gamma^1_{i1}\Gamma^1_{11} + \Gamma^2_{i1}\Gamma^1_{22} + \Gamma^3_{i1}\Gamma^1_{33}$. $\Gamma^1_{i1} = 0$ because $i \neq j$. $\Gamma^2_{i1} = 0$ because $i = 1, 3$ which is a

disappearing term, and $\Gamma^3_{i1}=0$ for $i=1,2$, and the whole term is zero.
$\Gamma^\alpha_{i2}\Gamma^2_{\alpha j}$: $\alpha\neq 0$ or else the first symbol vanishes. If $\alpha=1$ then $i=2$ for the first term to survive, but j must equal 2 for the second term to not equal zero. If $\alpha=2$ then $i=j=1$, however $i\neq j$. For $\alpha=3$, $i=j=3$, but $i\neq j$.
$\Gamma^3_{i3,j}$: only $i=2,3$ are nonvanishing. For $i=1$, $j=2,3$, but derivatives not with respect to r vanish. $i=2\Rightarrow j=1,3$ however then the function is only of θ and it vanishes.
$\Gamma^\alpha_{i3}\Gamma^3_{\alpha j}$: $\alpha\neq 0$. For $\alpha=1$, $i=j=3$ (not possible). For $\alpha=2$, $i=j=1$. If $\alpha=3$ then $i=1,2$, $j=2,1$, which does not vanish leaving $\Gamma^3_{13}\Gamma^3_{32}+\Gamma^3_{23}\Gamma^3_{31}$.
$-\Gamma^\alpha_{ij}\Gamma^3_{\alpha 3}$: This term can only have $\alpha=1,2$. $\alpha=1\Rightarrow i=j$ so it vanishes. For $\alpha=2$ the terms do not vanish leaving $-\Gamma^2_{12}\Gamma^3_{23}-\Gamma^2_{21}\Gamma^3_{23}$

The nonvanishing terms are

$$\Gamma^3_{13}\Gamma^3_{32}+\Gamma^3_{23}\Gamma^3_{31}-\Gamma^2_{12}\Gamma^3_{23}-\Gamma^2_{21}\Gamma^3_{23}=\frac{1}{r}\cot\theta+\frac{1}{r}\cot\theta-\frac{1}{r}\cot\theta-\frac{1}{r}\cot\theta=0$$

Therefore $R_{\mu\nu}=0$ for $\mu\neq\nu$. Solving for the diagonal terms:

$$\begin{aligned}R_{00}=&\Gamma^0_{00,0}-\Gamma^0_{00,0}+\Gamma^\alpha_{00}\Gamma^0_{\alpha 0}-\Gamma^\alpha_{00}\Gamma^0_{\alpha 0}\to 0+0\\&+\Gamma^1_{01,0}-\Gamma^1_{00,1}+\Gamma^\alpha_{01}\Gamma^1_{\alpha 0}-\Gamma^\alpha_{00}\Gamma^1_{\alpha 1}\to 0-\Gamma^1_{00,1}+\Gamma^0_{01}\Gamma^1_{00}-\Gamma^1_{00}\Gamma^1_{11}\\&+\Gamma^2_{02,0}-\Gamma^2_{00,2}+\Gamma^\alpha_{02}\Gamma^2_{\alpha 0}-\Gamma^\alpha_{00}\Gamma^2_{\alpha 2}\to 0-0+0-\Gamma^1_{00}\Gamma^2_{12}\\&+\Gamma^3_{03,0}-\Gamma^3_{00,3}+\Gamma^\alpha_{03}\Gamma^3_{\alpha 0}-\Gamma^\alpha_{00}\Gamma^3_{\alpha 3}\to 0-0+0-\Gamma^1_{00}\Gamma^3_{13}\end{aligned}$$

$$\begin{aligned}R_{00}&=-\Gamma^1_{00,1}+\Gamma^0_{01}\Gamma^1_{00}-\Gamma^1_{00}\Gamma^1_{11}-\Gamma^1_{00}\Gamma^2_{12}-\Gamma^1_{0i}\Gamma^3_{13}\\&=-\frac{U''}{2V}+\frac{U'}{4}\frac{V'}{V^2}+\frac{(U')^2}{4UV}-\frac{1}{r}\frac{U'}{V}\end{aligned}$$

$$\begin{aligned}R_{11}=&\Gamma^0_{10,1}-\Gamma^0_{11,0}+\Gamma^\alpha_{10}\Gamma^0_{\alpha 1}-\Gamma^\alpha_{11}\Gamma^0_{\alpha 0}\to\Gamma^0_{10,1}-0+\Gamma^0_{10}\Gamma^0 01-\Gamma^1_{11}\Gamma^0_{10}\\&+\Gamma^1_{11,1}-\Gamma^1_{11,1}+\Gamma^\alpha_{11}\Gamma^1_{\alpha 1}-\Gamma^\alpha_{11}\Gamma^1_{\alpha 1}\to 0+0\\&+\Gamma^2_{12,1}-\Gamma^2_{11,2}+\Gamma^\alpha_{12}\Gamma^2_{\alpha 1}-\Gamma^\alpha_{11}\Gamma^2_{\alpha 2}\to\Gamma^2_{12,1}-0+\Gamma^2_{12}\Gamma^2 21-\Gamma^1_{11}\Gamma^2_{12}\\&+\Gamma^3_{13,1}-\Gamma^3_{11,3}+\Gamma^\alpha_{13}\Gamma^3_{\alpha 1}-\Gamma^\alpha_{11}\Gamma^3_{\alpha 3}\to\Gamma^3_{13,1}-0+\Gamma^3_{13}\Gamma^3_{31}-\Gamma^1_{11}\Gamma^3_{13}\end{aligned}$$

$$
\begin{aligned}
R_{11} &= \Gamma^0_{10,1} + \Gamma^0_{10}\Gamma^0 01 - \Gamma^1_{11}\Gamma^0_{10} + \Gamma^2_{12,1} + \Gamma^2_{12}\Gamma^2 21 - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^3_{13,1} + \Gamma^3_{13}\Gamma^3_{31} - \Gamma^1_{11}\Gamma^3_{13} \\
&= \frac{U''}{2U} - \frac{(U')^2}{4U^2} - \frac{U'V'}{4UV} - \frac{V'}{Vr}
\end{aligned}
$$

$$
\begin{aligned}
R_{22} =& \Gamma^0_{20,2} - \Gamma^0_{22,0} + \Gamma^\alpha_{20}\Gamma^0_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^0_{\alpha 0} \to 0 - 0 + 0 - \Gamma^1_{22}\Gamma^0_{10} \\
&+ \Gamma^1_{21,2} - \Gamma^1_{22,1} + \Gamma^\alpha_{21}\Gamma^1_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^1_{\alpha 1} \to 0 - \Gamma^1_{22,1} + \Gamma^2_{21}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^1_{11} \\
&+ \Gamma^2_{22,2} - \Gamma^2_{22,2} + \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} \to 0 + 0 \\
&+ \Gamma^3_{23,2} - \Gamma^3_{22,3} + \Gamma^\alpha_{23}\Gamma^3_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^3_{\alpha 3} \to \Gamma^3_{23,2} - 0 + \Gamma^3_{23}\Gamma^3_{32} - \Gamma^1_{22}\Gamma^3_{13}
\end{aligned}
$$

$$
\begin{aligned}
R_{22} &= -\Gamma^1_{22}\Gamma^0_{10} - \Gamma^1_{22,1} + \Gamma^2_{21}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^1_{11} + \Gamma^3_{23,2} + \Gamma^3_{23}\Gamma^3_{32} - \Gamma^1_{22}\Gamma^3_{13} \\
&= \frac{rU'}{2UV} + \frac{1}{V} - \frac{rV'}{2V^2} - 1
\end{aligned}
$$

$$
\begin{aligned}
R_{33} =& \Gamma^0_{30,3} - \Gamma^0_{33,0} + \Gamma^\alpha_{30}\Gamma^0_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^0_{\alpha 0} \to 0 - 0 + 0 - \Gamma^1_{33}\Gamma^0_{10} \\
&+ \Gamma^1_{31,3} - \Gamma^1_{33,1} + \Gamma^\alpha_{31}\Gamma^1_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^1_{\alpha 1} \to 0 - \Gamma^1_{33,1} + \Gamma^3_{31}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^1_{11} \\
&+ \Gamma^2_{32,3} - \Gamma^2_{33,2} + \Gamma^\alpha_{32}\Gamma^2_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^2_{\alpha 2} \to 0 - \Gamma^2_{33,2} + \Gamma^3_{32}\Gamma^2_{33} - \Gamma^1_{33}\Gamma^2_{12} \\
&+ \Gamma^3_{33,3} - \Gamma^3_{33,3} + \Gamma^\alpha_{33}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^3_{\alpha 3} \to 0 + 0
\end{aligned}
$$

$$
\begin{aligned}
R_{33} &= -\Gamma^1_{33}\Gamma^0_{10} - \Gamma^1_{33,1} + \Gamma^3_{31}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^1_{11} - \Gamma^2_{33,2} + \Gamma^3_{32}\Gamma^2_{33} - \Gamma^1_{33}\Gamma^2_{12} \\
&= \left(\frac{rU'}{2UV} + \frac{1}{V} - \frac{rV'}{2V^2} - 1\right)\sin^2\theta = \sin^2\theta R_{22}
\end{aligned}
$$

4.4 The Ricci Scalar

The Ricci scalar is a contraction of the Ricci tensor;

$$R = R^{\mu}{}_{\mu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}\sin^2\theta R_{22}$$
$$= \frac{1}{U}R_{00} - \frac{1}{V}R_{11} - \frac{1}{r^2}R_{22} - \left(\frac{1}{r^2\sin^2\theta}\right)\sin^2\theta R_{22}$$
$$= \frac{1}{U}R_{00} - \frac{1}{V}R_{11} - \frac{2}{r^2}R_{22}$$
$$= -\frac{U''}{UV} + \frac{U'}{2}\frac{V'}{UV^2} + \frac{(U')^2}{2U^2V} - \frac{2}{r}\frac{U'}{UV} + \frac{2}{r}\frac{V'}{V^2} + \frac{2}{r^2}(1 - \frac{1}{V})$$

4.5 Substituting into the Einstein Equations

Using the metric and the variables as defined, the four following equations must be satisfied:

$$R_{00} - \frac{1}{2}g_{00}R = 0 = R_{00} - \frac{U}{2}R$$
$$0 = \frac{1}{r}\frac{V'}{V^2} + \frac{1}{r^2}(1 - \frac{1}{V})$$

$$R_{11} - \frac{1}{2}g_{11}R = 0 = R_{11} + \frac{V}{2}R$$
$$0 = -\frac{U'}{rUV} + \frac{1}{r^2}(1 - \frac{1}{V})$$

$$R_{22} - \frac{1}{2}g_{22}R = 0 = R_{22} + \frac{r^2}{2}R$$
$$0 = -\frac{U'}{U} + \frac{V'}{V} - \frac{rU''}{U} + \frac{rU'}{2}\frac{V'}{UV} + \frac{r(U')^2}{2U^2}$$

$$R_{33} - \frac{1}{2}g_{33}R = 0 = \sin^2\theta R_{22} + \frac{r^2\sin^2\theta}{2}R = R_{22} + \frac{r^2}{2}R$$

4.6 Solving and substituting into the metric

Notice $R_{00} - \frac{1}{2}Rg_{00} = 0$ is only in terms of V

$$
\begin{aligned}
0 &= \frac{V'}{V} + \frac{1}{r}(V-1) \\
-\frac{dr}{r} &= \frac{dV}{V(V-1)}
\end{aligned}
$$

Integrating both sides with the rule $\int \frac{dx}{ax+bx^2} = -\frac{1}{a}\ln\frac{a+bx}{x}$:

$$
\begin{aligned}
\ln r + C' &= \ln\frac{(V-1)}{V} \\
\frac{C}{r} &= \frac{V-1}{V} \\
\therefore V &= \frac{1}{1-\frac{C}{r}}
\end{aligned}
$$

Inserting this solution into $R_{11} - \frac{1}{2}g_{11}R = 0$

$$
\begin{aligned}
0 &= \frac{U'}{U}(1-\frac{C}{r}) - \frac{1}{r}(1-(1-\frac{C}{r})) \\
\frac{U'}{U} &= \frac{C}{r^2(1-\frac{C}{r})} = \frac{C}{r^2-Cr} \\
\frac{dU}{U} &= \frac{C\,dr}{r^2-Cr}
\end{aligned}
$$

Integrating both sides with the same formula as above:

$$\ln U = \frac{C}{C}\ln\frac{r-C}{r}$$

and exponentiating

$$U = 1 - \frac{C}{r}$$

This means that the Schwarzschild metric is

$$ds^2 = (1-\frac{2GM}{c^2 r})c^2\,dt^2 - \frac{dr^2}{(1-\frac{2GM}{c^2 r})} - r^2\,d\theta^2 - r^2\sin^2\theta\,d\psi^2$$

Where $C = \frac{2GM}{c^2}$.

5 Conclusion

These equations describe the space-time manifold around a point mass. This equation shows that there are singular points when $r = 0$ and when $r = \frac{2GM}{c^2}$; the first is the center of the black hole–no particle can be in exactly the same place as another. The second singular point represents the event horizon of a black hole, and the Schwarzschild radius–the radius at which a ball of mass M collapses into a black hole. This metric also shows that the black hole consists of two connected sets of definition, that within the black hole, and the area outside. The singularity at $\frac{2GM}{c^2}$ can be removed using a different coordinatization (Kruskal-Szekeres coordinates explained more in [7]), which gives the maximally extended solution and shows that space inside and outside the black hole are not completely different regions. Other solutions expanding on the Schwarzschild derivation account for charge (Reissner-Nordström solution) and rotating black holes (Kerr-Newman). The really troubling component is that the only assumption of spherically symmetric space produced a curvature singularity: this baffled scientists of the time and Schwarzschild's solution was thought to be wrong. However, it holds still today and is used to investigate the curvature around black holes, and even the fact that curvature like the equations described exists in some form in nature is astounding–it's the inspiration for many a physicist to continue working to uncover other mysteries of nature.

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