An Introduction to Tropical Geometry

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1 Introduction

In their paper "A Bit of Tropical Geometry" [1], Erwan Brugallé and Kristin Shaw present an elementary introduction to the theory of tropical geometry in the plane. The authors explicitly state that they intend the paper to be accessible to first-year students of mathematics.

Tropical geometry arises from the study of an algebra over the real numbers in which the usual product is replaced with the sum and the usual sum is replaced with the maximum. Polynomials defined using these operations are convex and piecewise linear functions. Tropical algebraic curves can be associated to tropical polynomials in two variables and, because of the combinatorial simplicity of tropical polynomials, are much easier to study and understand than classical algebraic curves.

In fact, most concepts from classical algebraic geometry have tropical analogues. This would be little more than a neat mathematical coincidence if it were not also the case that classical objects (lines, polynomials, and curves) can be transformed into tropical objects while preserving many of those characteristics, and vice versa. Constructing classical algebraic curves with given properties is much more difficult than constructing a tropical curve with the desired properties and then transferring it into the classical world.

The paper [1] introduces tropical algebra, tropical algebraic curves in the plane, the intersection theory thereof, and the "patchworking" process that turns tropical curves into classical ones. It closes with some sketches of current directions in tropical research. This paper roughly follows that trajectory, pausing to reproduce the proof of a tropical analogue to Bézout's theorem (which is itself a result in the intersection theory of classical algebraic curves).

A note on nomenclature: "tropical geometry" was initially known as "max-plus geometry". The current name was bestowed upon the subject by a group of French computer-science researchers in honor of their colleague Imre Simon, who did work on max-plus geometry (as it was known at the time) in Brazil. The name has little to do with the subject itself and much more to do with the French perspective on Brazil.

2 Tropical algebra

Tropical geometry by nature requires a good deal of algebraic definitions before any sort of forward mathematical motion can happen. Throughout these definitions and the rest of this paper, tropical algebraic expressions and operations are distinguished from their classical counterparts by enclosing them in quotes, like this: "$x + yz$". This is the convention used throughout [1].

2.1 The tropical semi-field

A semi-field is a field without additive inverses. None of our development of tropical geometry requires advanced algebraic machinery, but the semi-field rules are important and we present them here for later use.

Definition 1. A *semi-field* is a set S equipped with two binary operations $\times$ (multiplication) and $+$ (addition), obeying the following rules:

1. The operations are *closed*: For every pair a, b of elements of S, both $a \times b$ and $a + b$ are members of S.
2. The operations are *associative*: For every $a, b, c \in S$, the equations $(a \times b) \times c = a \times (b \times c)$ and $(a + b) + c = a + (b + c)$ hold.
3. Each operation has an *identity element*: there exists an element e of S such that $a \times e = a$ for all elements a of S, and there exists an element o of S such that $a + o = a$ for all elements a of S.
4. The operations *commute*: For all a, b in S, the equations $a \times b = b \times a$ and $a + b = b + a$ hold.
5. There exist *multiplicative inverses*: for every element a of S aside from o, there exists an element $a^{-1} \in S$ such that $a \times a^{-1} = e$.
6. Multiplication *distributes* over addition: for all a, b, c in S, $a \times (b + c) = a \times b + a \times c$.

Tropical algebra takes place in one particular instance of the above.

Definition 2. The *tropical semi-field*, denoted by $\mathbb{T}$, is the set of tropical numbers $\mathbb{R} \cup \{-\infty\}$ equipped with the commutative operations "$a + b$" $= \max(a, b)$ and "$a \times b$" $=$ "ab" $= a + b$ for any pair a, b of tropical numbers. The tropical sum of $-\infty$ and any tropical number x is x, and the tropical product of $-\infty$ and any tropical number is $-\infty$.

The reader is encouraged to verify that the semi-field rules hold in $\mathbb{T}$. Tropical multiplication "$a \times b$" may be denoted "ab". Integer exponents denote repeated multiplication, so "a^4" $=$ "$aaaa$", and the familiar sigma notation for finite sums works as one would expect:

$$\text{``}\sum_{j=0}^{3} a_j = a_0 + a_1 + a_2 + a_3\text{''}.$$

Semi-fields, like many other algebraic structures, can be found in the real number system: the set $\mathbb{R}_{\geq 0}$ of nonnegative reals forms a semi-field under its usual addition and multiplication. Suggestively, $\mathbb{R}_{\geq 0}$ is exactly the preimage of $\mathbb{T}$ under the map $x \mapsto \log x$ if we define $\log(0) = -\infty$. This will be further explored in Section 5. Though both $\mathbb{R}_{\geq 0}$ and $\mathbb{T}$ are semi-fields, they are algebraically very different. Rather obviously, we can turn $\mathbb{R}_{\geq 0}$ into a field by adding elements defined to be the unique additive inverses of their nonnegative counterparts. These are the familiar negative real numbers. Suppose we do the same with $\mathbb{T}$ by introducing a symbol $\neg$ and defining "$a + \neg a$" $= -\infty$, we have a problem. Observe that addition in $\mathbb{T}$ is idempotent, unlike real addition: "$a + a$" $= a$. Hence, "$(a + a) + \neg a$" $=$ "$a + \neg a$" $= -\infty$. However, "$a + (a + \neg a)$" $=$ "$a + (-\infty)$" $= a$. Evidently, introducing additive inverses to $\mathbb{T}$ does away with associativity, and we have actually worsened the algebraic situation! Thus $\mathbb{T}$ is *really* a semi-field, unlike $\mathbb{R}_{\geq 0}$.

2.2 Tropical polynomials

As in classical algebraic geometry, the main objects of study in tropical geometry are polynomials and the geometric locations of their roots. Of course, tropical polynomials (and their roots) are markedly different from their classical counterparts.

Definition 3. A *tropical polynomial* of degree d is a function $p : \mathbb{T} \to \mathbb{T}$ of the form

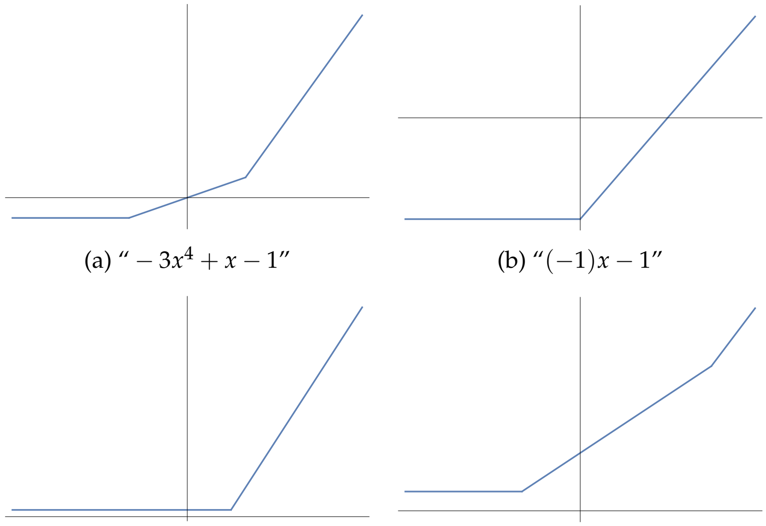
$$p(t) = \text{``}\sum_{i=0}^{d} a_i t^i\text{''} = \max_{i=0}^{d}(it + a_i),$$

where the coefficients a_i are real numbers and d is a natural number.

Classically, a real polynomial $q(x)$ has a root at some point x_0 when $q(x_0) = 0$. In tropical algebra, this translates to finding a x_0 for a tropical polynomial $p(x)$ such that $p(x_0) = -\infty$. Since there are no additive inverses in $\mathbb{T}$, this does not really work: the only polynomial that would have a zero in this sense would be the constant function $p(x) = -\infty$.

However, we can look to the fundamental theorem of algebra and find a more generalizable notion of "root", namely that of factorability. A polynomial $p(x)$ has a root at x_0 when it can be factored into $p(x) = (x - x_0)^k q(x)$, where $q(x)$ is another polynomial with no root at x_0. In tropical geometry, this definition makes sense, albeit with a plus in the factored term because of the absence of tropical negatives.

Definition 4. A point $x_0 \in \mathbb{T}$ is a tropical root of order at least k of a tropical polynomial $p(x)$ if there exists another tropical polynomial $q(x)$ such that $p(x) =$ "$(x + x_0)^k q(x)$" for some k. The largest k for which this is possible is the multiplicity of the root x_0.



(a) "$-3x^4 + x - 1$" (b) "$(-1)x - 1$"

(c) "$x^2 + 1$" (d) ""

Figure 1: Plots of some tropical polynomials. The axes are scaled oddly, especially in (c), and I'm unsure how to convince Mathematica to not do that.

Geometrically, the roots of a tropical polynomial are exactly the points at which the graph has corners. The multiplicity of a root is the difference between the slopes of the two segments meeting at the root's corresponding corner. Several plots of tropical polynomials may be found in Figure 1.

If the geometric argument is not compelling, note that with this definition of root the tropical numbers are algebraically closed. The proof of the following proposition is left as an exercise to the reader in [1].

Proposition 1. *A tropical polynomial of degree d has exactly d roots when counting with multiplicities.*

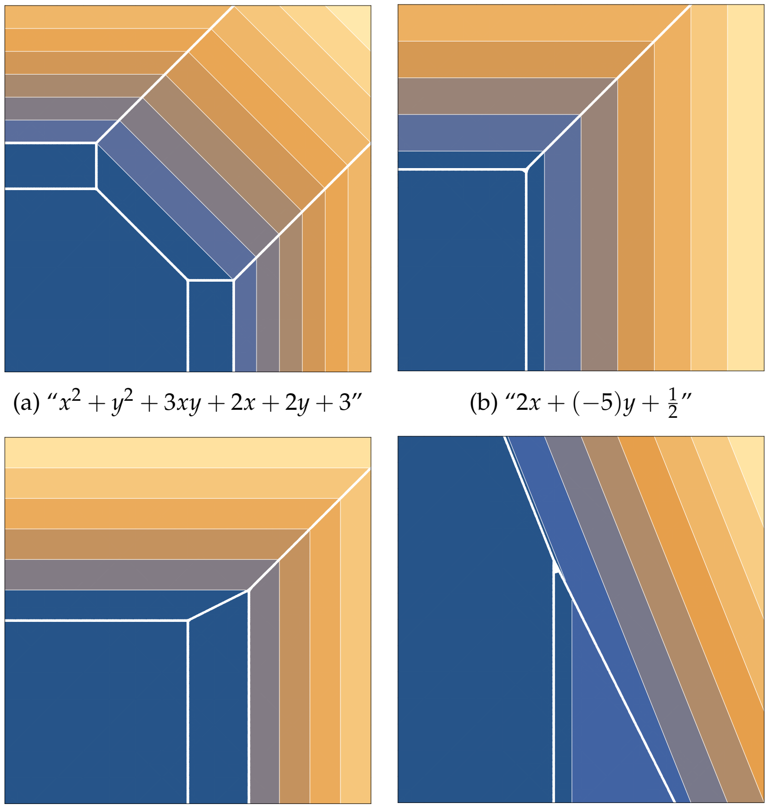
3 Tropical curves

3.1 Definitions

Real algebraic curves in the plane are the zero sets of polynomials in two variables. In the tropics, we can define a similar notion. First, some book-keeping: we need to add another variable to our polynomials.

Definition 5. A tropical polynomial of degree d in two variables is a function $p(x, y)$ of the form

$$p(x,y) = \text{``}\sum_{i,j} a_{i,j} x^i y^i\text{''} = \max_{i,j}(ix + jy + a_{i,j}),$$



(a) “$x^2 + y^2 + 3xy + 2x + 2y + 3$” (b) “$2x + (-5)y + \frac{1}{2}$”

(c) “$(-1)x^2 + y^2 + x + 0$” (d) “$x^5y^2 + \frac{1}{2}x - 1$”

Figure 2: Contour plots of some tropical polynomials in two variables.

where i and j are integers ranging from 0 to d that satisfy $i + j \leq d$ in each term of the polynomial.

Like polynomials in one variable, tropical polynomials in two variables are convex piecewise linear functions. The curve defined by a tropical polynomial in two variables is the set of points in $\mathbb{R}^2$ at which the polynomial has ”corners”.

Definition 6. Let $P(x, y) =$ “$\sum_{i,j} a_{i,j} x^i y^j$” be a tropical polynomial. The tropical curve C defined by $P(x, y)$ is the set of points (x_0, y_0) in $\mathbb{R}^2$ such that there exist pairs $(i, j) \neq (k, j)$ satisfying $P(x_0, y_0) = a_{i,j} + ix_0 + jy_0 = a_{k,l} + kx_0 + ly_0$.

Several contour plots of tropical polynomials in two variables may be found in Figure 2. Their associated tropical curves are the thick white lines at which the contours meet in angles. From the figure it should be clear that tropical curves always consist of a finite number of line segments (or rays), which we will call *edges*, meeting in points which we call *vertices*. Brugallé and Shaw give a detailed discussion of this in [1].

The weight of an edge of a tropical curve is roughly analogous to the multiplicity of a root of a tropical polynomial in one variable.

Definition 7. The weight w_e of an edge e is the maximum of the greatest common divisors of the numbers $|i-k|$ and $|j-l|$ for all pairs (i,j) and (k,j) that correspond to the edge. Formally,

$$w_e = \max_{M_e}(\gcd(|i-k|, |j-l|)),$$

where

$$M_e = \left\{(i,j),(k,l) : \forall x_0 \in e, P(x_0,y_0) = a_{i,j} + ix_0 + jy_0 = a_{k,l} + kx_0 + ly_0\right\}.$$

3.2 Dual subdivisions

If this talk of edges and vertices sounds vaguely graph-theoretic, that's because it is. While tropical curves have unbounded edges and hence are not obviously graphs in any familiar sense, we can produce planar graphs known as *dual subdivisions* that aid in our study of curves.

Suppose we have a tropical polynomial $p(x,y)$ of degree d with coefficients $a_{i,j}$. It defines a curve C in $\mathbb{R}^2$. At each point of C, at least two monomials of $p(x,y)$ are equal to one another and greater than the other monomials of $p(x,y)$. All the points $(i,j) \in \mathbb{Z}^2$ such that $a_{i,j} \neq -\infty$ are contained in the triangle Δ_d with vertices $(0,0)$, $(0,d)$, and $(d,0)$. For simplicity, we suppose throughout the remainder of this text that all our polynomials have $a_{0,0}$, $a_{d,0}$, and $a_{0,d}$ not equal to $-\infty$.

Definition 8. Given a finite set of points A in $\mathbb{R}^2$, the *convex hull* of A is the unique convex polygon with vertices in A that also contains A.

Therefore, the triangle Δ_d is the convex hull of the points (i,j) at which $a_{i,j} \neq -\infty$.

Definition 9. Suppose we have a curve C defined by a tropical polynomial $p(x,y)$. The *dual triangle* associated to a vertex $v = (x_0,y_0)$ of C, denoted Δ_v, is the convex hull of the points $(i,j) \in \mathbb{Z}^2$ at which $p(x_0,y_0) = a_{i,j} = ix_0 + jy_0$.

The convex piecewise linearity of $p(x,y)$ implies that the set of Δ_v for all vertices v of C forms a subdivision of Δ_d.

Definition 10. The *dual subdivision* of a curve C defined by a polynomial $p(x,y)$ of degree d is the union of the triangles Δ_v for each vertex v of C.

4 Tropical intersection theory

In classical geometry, Bézout's theorem states that two algebraic curves in the plane of degrees d_1 and d_2, respectively, intersect in exactly $d_1 d_2$ points. This is a projective theorem: two parallel lines do not intersect in $\mathbb{R}^2$, but they do intersect "at infinity". In tropical geometry, if we define notions of intersection and multiplicity carefully, we can get an analogous theorem for the intersections of tropical curves.

The analogue for tropical curves is the following theorem, attributed to B. Strumfels. Note that, as in classical algebraic geometry, the union $C_1 \cup C_2$ of two tropical curves C_1 and C_2 defined by polynomials $p_1(x,y)$ and $p_2(x,y)$ is itself a tropical curve defined by the polynomial $q(x,y) = "p_1(x,y)p_2(x,y)"$.

Theorem 1. *Let C_1 and C_2 be two tropical curves of degrees d_1 and d_2, respectively, intersecting in a finite number of points away from the vertices of the two curves. Then the sum of the tropical multiplicities of all points in the intersection of C_1 and C_2 is equal to $d_1 d_2$.*

Proof. Let s be the sum of the multiplicities of the points in the intersection. Consider the dual subdivision of the tropical curve $C_1 \cup C_2$. The polygons of the subdivision fall into three (distinct, because intersections occur away from vertices) categories: those dual to a vertex of C_1 having total area $\frac{1}{2}d_1^2$, those dual to a vertex of C_2 having total area $\frac{1}{2}d_2^2$, and those dual to an intersection point of C_1 and C_2. Since the curve $C_1 \cup C_2$ is of degree $d_1 + d_2$, the sums of the areas of these polygons is equal to th area of $\Delta_{d_1+d_2}$, which is $\frac{1}{2}(d_1+d_2)^2$. Therefore

$$s = \frac{(d_1+d_2)^2 - d_1^2 - d_2^2}{2} = d_1 d_2. \qquad \square$$

5 Moving between tropical and classical geometry

5.1 Dequantization

Classical curves can be degenerated into tropical curves using a process known as Maslow dequantization. Brugallé and Shaw sum it up nicely in [1] when they write that "*tropical geometry is the image of classical geometry under the logarithm with base $+\infty$.*"

Let us return to the notion of semi-field, and the example of $\mathbb{R}_{\geq 0}$. If we take $t > 0$, the logarithm of base t provides a bijection between the sets $\mathbb{R}$ and $\mathbb{T}$. We can define a semi-field structure on $\mathbb{T}$ with this bijection and the existing structure on $\mathbb{R}_{\geq 0}$ with the operations " $+_t$ " and " $\times_t$ ", defined by

$$"x +_t y" = \log_t(t^x + t^y) \quad \text{and} \quad "x \times_t y" = log_t(t^x t^y) = x + y.$$

If we take $t \to \infty$ these operations are exactly the familiar tropical addition and multiplication!

We can *dequantize* curves by applying a sort of componentwise logarithm of base t to their points, mapping $(x,y) \mapsto (\log_t |x|, \log_t |y|)$.

5.2 Patchworking

Planar algebraic curves of low degree are relatively well-understood objects. In degree two, they are the conics, in degree three they are the cubics, and so on. These curves consist of bounded and unbounded connected components in the plane which are *arranged* in a certain way, i.e. sometimes two connected components will be contained in one another and sometimes they will not. The question of arrangement is a topological one.

Two curves have the same arrangement if their connected components are nested within one another in the same way.

David Hilbert's 16th problem, delivered among his other 23 famous problems to the International Congress in Mathematics at Paris in 1900, asks mathematicians to establish all possible arrangements of real algebraic curves. This problem has been solved for conics since antiquity, and at Hilbert's time was completely solved for curves of degree up to 4.

Many of the modern advances in this problem happened before the introduction of tropical geometry, but Oleg Viro's powerful method of patchworking turns out to have a natural expression in the tropics.

Patchworking was developed by Viro in order to construct algebraic curves of a given degree and arrangement. The procedure is largely combinatorial, and proceeds in four steps. First, produce a tropical curve. Then, with some constraints, erase some of its edges. What remains is still piecewise linear and is known as a real tropical curve. The configuration of the real tropical curve corresponds to the configuration of a real algebraic curve.

Theorem 2. *Given any real tropical curve of degree d, there exists a real algebraic curve of degree d with the same arrangement.*

This is a powerful tool. While real tropical curves of a given arrangement require some experience to construct, their construction is a purely discrete, combinatorial process which is much simpler than the world of real algebraic curves.

References

[1] Erwan Brugallé and Kristin Shaw. A bit of tropical geometry. *The American Mathematical Monthly*, 121(7):pp. 563–589, 2014.