The Two Envelopes Problem and Utility Theory

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1 Introduction

We pose the following problem: Imagine that you are presented with two envelopes containing a positive amount of money. However, one of the envelopes contains an amount twice as much as the other. After picking an envelope and inspecting the contents, you are given a choice between keeping the contents or exchanging it for the other envelope. We discuss the paradoxical nature of a mathematical calculation of the solution, and then add the assumption that we have knowledge of the conditional probability that the envelope you picked contained more money than the remaining envelope, given that you have inspected the contents. This assumption allows us to calculate a *General Exchange Condition* under which you are able to determine whether or not it would be in your best interest to switch envelopes (assuming that you wish to maximize your return). We then discuss the relationship of monetary returns to utility theory, and examine an assumption posed by the domain of mathematical economics regarding boundedness.

2 Definitions

Definition A **random variable** is a variable that may take on a set of possible values (either discrete or continuous) with an associated probability of taking on each possible value (this is a somewhat informal definition, but sufficient for the scope of the paper).

Definition Given a discrete random variable X, the **expected value** is given by $\sum_{i=1}^{\infty} x_i p_i$ where the x_i are the possible values X may assume and the p_i are the related probabilities of assuming those values. if X is continuous, then the **expected value** is given by $\int_{-\infty}^{\infty} x f(x) dx$ where $f(x)$ is a function that denotes the probability that X assumes the value x.

Definition A person's **utility** refers to their total satisfaction from a good or service.

Definition A **conditional probability** gives the probability of an event, given that another event has occurred. For example, if we denote A and B as two events, then $Pr(A|B)$ is the conditional probability of event A, given that the event B has occurred.

Definition **Bayes' theorem** states that if $A_1, A_2, ..., A_j$ are mutually exclusive events that form the set of all possible outcomes of an experiment, then

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)}$$

Definition A **probability distribution** is a function that assigns a probability to all possible outcomes of a random variable.

Definition The **cumulative distribution function** of a random variable X is given by $F_X(x) = P(X \leq x)$. That is, given some value x, the cumulative distribution function the random variable X is the probability that it assumes a value less than or equal to x.

Definition If X is a continuous random variable, then the **probability density function**, denoted as $f(x)$, is such that for some values a and b,

$$Pr(a \leq X \leq b) = \int_a^b f(x) dx$$

3 The Paradoxical Solution

If we do not assume knowledge of any **conditional probabilities**, one might take a probabilistic approach of deciding whether to switch or not. If we open our envelope and see that it contains \$$x$, then there is a $\frac{1}{2}$ probability that the other envelope contains \$$2x$ and a $\frac{1}{2}$ probability it contains \$$\frac{x}{2}$. If we calculate the expected value of switching, we would find that

$$
\begin{aligned}
E[X] &= \frac{1}{2}(2x) + \frac{1}{2}\left(\frac{x}{2}\right) \\
&= \frac{5}{4}x
\end{aligned}
$$

where X is a **random variable** denoting the dollar amount received from the other envelope. We then find that we should expect to gain more from switching envelopes. But this result is puzzling, as it implies that regardless of the amount found in the first envelope, we should always decide to switch envelopes. It suggests that the information about the amount of money in the envelope we have picked was not necessary, and we will essentially pick an envelope at random with the expectation that it was the best possible decision. To address this issue, we add the assumption that information about the amount of money in the envelope we picked is useful in determining the probability that the other envelope contains more money, conditioning on the fact that we have observed the contents of the first envelope.

4 Deriving the Exchange Condition

Let us denote the **random variables** L as the larger amount of money of the two envelopes, and S as the smaller of the two amounts, where L and S are positive. Then see that $S = \frac{L}{2}$, so we may write that

$$
Pr(L \leq x) = Pr\left(S \leq \frac{x}{2}\right)
$$

where x $\in (0, \infty)$. Denote the first envelope we pick as E_1 and the other envelope as E_2. Additionally, denote the amount contained in E_1 as the **random variable** X and the amount contained in E_2 as the **random variable** Y. Because we pick E_1 at random, the probability that X is equal to L is the same as the probability that X is equal to S. Upon opening envelope E_1, we observe the contents and see that $X = x$. Then the condition under which we should decide to switch envelopes is

$$
E[Y|X = x] = \left(\frac{x}{2}\right)Pr(X = L|X = x) + (2x)Pr(X = S|X = x) > x
$$

or equivalently,

$$
\left(\frac{1}{2}\right)Pr(X = L|X = x) + (2)Pr(X = S|X = x) > 1
$$

Note that $X \neq L$ since we wish to determine the condition under which to switch envelopes, which implies that $X = S$. See that

$$Pr(X = S|X = x) = 1 - Pr(X = L|X = x)$$

These two expressions then yield

$$\left(\frac{1}{2}\right)Pr(X = L|X = x) + 2[1 - Pr(X = L|X = x)] > 1$$

or equivalently,

$$\left(\frac{3}{2}\right)Pr(X = L|X = x) < 1$$

And so we denote the *General Exchange Condition* as

$$Pr(X = L|X = x) < \frac{2}{3} \tag{1}$$

We conclude that given knowledge of the underlying **probability distribution** and the observed value of $X = x$, we should switch envelopes if and only if the *General Exchange Condition* is satisfied [1].

5 Discrete and Continuous Probability Distributions

In this section, we derive specific conditions under which to switch envelopes when the distribution of dollar amounts in the envelopes assume either discrete or continuous values.

5.1 Discrete Distributions

In the discrete case, the **random variable** may only assume a countable number of distinct values. So let X, L, and S be discrete **random variables**. Now we make the assumption that there exists some fixed $m \in \mathbb{Z}^+$ such that the amount of money in the chosen envelope can be written as $\$2^k m$ for some $k \in \mathbb{Z}$. We denote the prize values in this way so that it facilitates the expression of dollar amount contained in the other unknown envelope as either half or double the value in the chosen envelope. Then we define the **probability distribution**

$$p_k = Pr(X = 2^k m|X = L)$$

where $k \in \mathbb{Z}$. Since we want that $\{..., p_{-1}, p_0, p_1, ...\}$ defines a **probability distribution**, assume that $p_k \geq 0$ and $\sum_{-\infty}^{\infty} p_k = 1$. Recall that since $S = \frac{L}{2}$, it follows that

$$p_{k+1} = Pr(X = 2^{k+1}m|X = L) = Pr(X = 2^k m|X = S)$$

Now suppose we play the game with this knowledge, and pick an envelope containing the amount $X = 2^k m$. Now having observed the amount inside, see that

$$p_k + p_{k+1} = Pr(X = 2^k m|X = L) + Pr(X = 2^k m|X = S) > 0$$

We then use **Bayes' Theorem** to show that

$$\begin{aligned} Pr(X = L|X = 2^k m) &= \frac{Pr(X = 2^k m|X = L)Pr(X = L)}{Pr(L = 2^k m|X = L)Pr(X = L) + Pr(S = 2^k m|X = S)Pr(X = S)} \\ &= \frac{p_k}{p_k + p_{k+1}} \end{aligned}$$

Then by using the *General Exchange Condition*, we can show that the *Exchange Condition for Discrete Distributions* is

$$\frac{p_k}{p_k + p_{k+1}} < \frac{2}{3}$$

or equivalently,

$$p_k < 2p_{k+1} \tag{2}$$

This result implies that it is profitable to switch envelopes if and only if the unconditional probability that of picking the "larger" envelope is less than twice the unconditional probability the "smaller" envelope [1].

5.2 Continuous Distributions

Now in the continuous case, the **random variables** X, L, and S assume an uncountable infinite number of values. Recall that

$$Pr(S \leq x) = Pr(L \leq 2x)$$

We can also see that the **cumulative distribution functions** of L and S are such that

$$F_S(x) = F_L(2x)$$

Differentiating a **cumulative distribution function** yields its respective **probability density function**

$$f_S(x) = 2f_L(2x)$$

To return to our question at hand, suppose that we pick envelope E_1 and observe that X satisfies the expression $x \leq X \leq x + dx$. Then given this observation, the **conditional probability** that E_1 is the "larger" envelope is analogous to the discrete case, and

$$\begin{aligned} Pr(X = L | x \leq X \leq X + dx) &= \frac{f_L(x)dx}{f_L(x)dx + f_S(x)dx} \\ &= \frac{f_L(x)}{f_L(x) + 2f_L(2x)} \end{aligned}$$

So similarly, we will find that the *Exchange Condition for Continuous Distributions* is given by

$$\frac{f_L(x)}{f_L(x) + 2f_L(2x)} \leq \frac{2}{3}$$

or equivalently,

$$f_L(x) < 4f_L(2x) \tag{3}$$

We thus conclude that switching is profitable if and only if the unconditional density of L at x (or the observed value of X) is less than four times the unconditional density of L at twice the observed value of X [1].

6 Utility Theory

6.1 Boundedness of Utility

Let us return to the original statement of the problem in which we came to the paradoxical solution of always choosing to switch envelopes. Instead of money, we instead replace the prize of the envelopes with **utility**. We assert that if it is true that the best choice is to always switch, then our utility must be unbounded. We show this by contradiction. If we suppose that utility is bounded, then there exists a number c such that the value L now denoting the **utility** of the "larger" envelope is such that

$$Pr(L > c) = 0, Pr\big(\frac{c}{2} < L \leq c\big) > 0$$

but then see that

$$Pr(X = L | \frac{c}{2} < L \leq c\big) = 1$$

which implies that $Pr(\frac{c}{2} < L \leq c) > 0$, and the *General Exchange Condition* would fail with positive probability. In order for switching envelopes to always be the profitable solution, it must be that the utility of the other envelope is unbounded. However, it is puzzling to consider that there exists an envelope yielding an unbounded utility gain. In an attempt to propose a solution to this issue, mathematician Daniel Bernoulli developed key ideas in the area of utility theory [2]. Bernoulli states that the value of an item does not necessarily stem from its price, but rather its utility it yields to a specific person. That is, the price of an object may be determined taking into consideration only the object itself, but the utility of the object depends upon the circumstances of the person evaluating the object [2, pg 24]. In order to briefly demonstrate this, Bernoulli gives the example that a poor man would receive more utility from an item than a rich man, and that a rich prisoner who needs two thousand more ducats (early European currency) to be bailed from prison would receive more utility from gaining this amount than a poor prisoner who might need four thousand ducats [2, pg 25]. In addition, he claims that any amount of increase in wealth will always result in an increase in utility inversely proportionate to the quantity of goods already possessed [2, pg 25]. Furthermore, utility is mathematically modeled as a logarithmic curve as a function of a person's wealth. Doing so incorporates the characteristic of diminishing returns to utility, yielding a bound to the proposed utility function. If we assume that utility is bounded, we can then see that it is not logical to always switch envelopes in the original statement of the two envelope problem.

6.2 St. Petersburg Paradox

Say we have the following problem: Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation. So, letting X denote a discrete **random variable** which describes Paul's yield, it is easy to see that

$$\begin{aligned} E[X] &= (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{4}\right) + (4)\left(\frac{1}{8}\right) + ... \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... \\ &= \infty \end{aligned}$$

Given this expected winnings, is it reasonable for Peter to charge Paul twenty ducats as fee for playing the game? Around the same time that Bernoulli wrote on utility theory, mathematician Gabriel Cramer studied the problem and explained that it seemed absurd to charge this large fee for playing the game [2, pg 33]. In order to examine this discrepancy, Cramer explains that "in theory, mathematicians evaluate money in proportion to its quantity while, in practice, people with common sense evaluate money in proportion to the utility they can obtain from it" [2, pg 33]. Although our monetary gain would double at each successful throw, according to our theory of utility, we would not expect Paul's utility to always double as the number of successful throws approaches infinity. Problems such as the Two Envelopes Problem and the St. Petersburg Paradox highlighted a conflict between mathematical theory and actual practice, leading to a development of utility and risk theory by mathematicians and economists.

References

[1] S. J. Brams and D. M. Kilgour, The Box Problem: To Switch or Not to Switch, *Mathematics Magazine*, vol. 68, No. 1 (Feb., 1995), 27-34

[2] D. Bernoulli, Exposition of a new theory on the measurement of risk (English translation of 1738 article), *Econometrica 22* (1954), 23-36