Easy Maximum Principle

Theorem 1. *Suppose u is bounded and harmonic in a bounded open set Ω. Suppose $\limsup\limits_{z \to p \in \partial\Omega} u(z) \leq M$ for all $p \in \partial\Omega$. Then $u(z) \leq M$.*

Proof. Let $\delta > 0$. Let $W = \{z : z \in \Omega, u(z) > M + \delta\} \subset \Omega$. Then $\overline{W} \subset \overline{\Omega}$, so $\partial W \subset \Omega \cup \partial\Omega$. First we prove that $\partial W \cap \partial\Omega = \emptyset$. For if $p \in \partial\Omega$ then $\limsup_{z \to p \in \partial\Omega} u(z) \leq M$. But if $p \in \partial W$ then $\limsup_{z \to p \in \partial W} u(z) \geq M + \delta$. So $\overline{W} \subset \Omega$ and u is continuous on $\overline{W}$. Now a point p in ∂W must contain points in W and points not in W. Since u is continuous, if $u(p) > M + \delta$, so are nearby points (W is open). So it must be the case that $u(p) \leq M + \delta$. In other words $u(p) \leq M + \delta$ for points $p \in \partial W$. By the maximum principle $u(p) \leq M + \delta$ for all $p \in W$. This contradicts the definition of W. So $W = \emptyset$. In other words $u(p) \leq M + \delta$ for all $p \in \Omega$. Since $\delta > 0$ is arbitrary, $u(p) \leq M$ for all $p \in \Omega$.