Article Review: A Survey of Geometric Calculus and Geometric Algebra

Josh Pollock

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1 Introduction

In his article *A Survey of Geometric Calculus and Geometric Algebra* [4], Professor Alan Macdonald provides a brief introduction to geometric algebra (GA) and geometric calculus (GC) along with some applications to physics and a brief mention of the related projective and conformal geometric algebras. He only expects the reader to have knowledge of linear algebra and vector calculus. In this review, I hope to whet your appetite for GA and GC by showing some of its important results.

Geometric algebra is a relatively recent addition to the family of algebras used to reason about space. In class we have already encountered the most common: vectors, matrices, and complex numbers. We have also briefly dealt with differential forms, which are particularly useful for concisely expressing fundamental theorems of algebra such as Green's and Stokes' theorems. Unfortunately, while conceptually simpler than other tools for working with n-dimensional spaces, the aforementioned algebras are cumbersome for talking about objects such as planes and volume elements. Luckily there is another class of algebras that are better suited to this task. Some notable examples of this latter group are tensor, Grassmann, and Clifford algebras. Geometric algebra extends this second lineage, adding explicit geometric significance to the often abstractly-presented Clifford algebra.

According to Professor David Hestenes, the creator of geometric algebra,[1] GA's foundation is the notion of a directed number [3]. The notion of a vector provides a reasonable first approximation to this idea since it has both a magnitude and a direction; however, we shall see that restricting the idea of a directed number to one-dimensional objects restricts our ability to think about n-dimensional objects in a unified way.

Geometric algebra is built around two key generalizations of vector algebra: multivectors and the geometric product. Multivectors are generalizations of vectors to higher dimensions. The geometric product (roughly) incorporates the dot and cross products into a unified product, and then generalizes this to multiplication on arbitrary multivectors. This product was first introduced by Hermann Grassmann in 1877, although it remained relatively unknown until sometime later when William Clifford, who built on Grassmann's widely published work, rediscovered it [3].

Geometric calculus builds on the foundation of geometric algebra. It allows one to use the language of GA to do multivariable calculus in simpler ways, and often makes it possible to derive formulas without introducing an explicit coordinate system. A particularly noteworthy result in GC is the generalized Fundamental Theorem of Calculus, and one of the goals of this paper is for the reader to have a rudimentary understanding of the statement of the theorem. This theorem encompasses Green's, Stokes', the divergence, Cauchy's, and the residue theorems into a single notational framework.[2] In doing so, a relationship

[1] The name geometric algebra may have first been associated with Clifford algebras by Emil Artin in 1957 [1]; however, his treatment of the subject is slightly different.

[2] A similar result can be obtained via differential forms known as the generalized Stokes'

between the gradient, divergence, and curl operators can be drawn that is not as easily seen in conventional undergraduate analysis approaches.

2 Motivating the Geometric Product and Higher Dimensional Geometric Objects

In this section I hope to convince you that the traditional treatment of vector algebra and calculus is incomplete.

2.1 Products

We are already familiar with the two vector products $\cdot$ and $\times$. While useful, these products are actually not very well behaved.

We first consider the cross product. Although it distributes nicely over addition, it is not associative! It is left as an exercise for the reader to find three vectors a, b, c such that $a \times (b \times c) \neq (a \times b) \times c$. Moreover, the cross product generalizes poorly to dimensions higher than 3.

Recall that $a \times b$ is orthogonal to a and b. Already this poses some difficulty in $\mathbb{R}^3$, because there are essentially two orthogonal directions to choose (this results in the right hand rule). However, if we consider dimension 4, for example, there is an entire plane of vectors orthogonal to a and b. In fact in dimension n, if we wish to extend the cross product "naturally", we will end up with an $n-2$-dimensional object. This seems like a backwards way to describe the product of two vectors.

The dot product does not fair any better. While $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, allowing us to multiply arbitrarily many vectors, the same cannot be said for the dot product. Perhaps, one might say, we could extend the definition of a dot product to mean scalar multiplication in the case that we try to take the dot product of a scalar and a vector. I.e. we might want to let $a \cdot \mathbf{v} = \mathbf{v} \cdot a \equiv a\mathbf{v}$ for any $a \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$. Unfortunately this product is not associative either! Again, this is left as an exercise for the reader.[3]

The two ubiquitous vector products, $\cdot$ and $\times$, encapsulate much of the "measurement" we wish to do with vectors. In particular, the dot product is used to measure distance and in some sense captures the "parallel" component of two vectors. It is maximal when its inputs are parallel and 0 when its inputs are orthogonal. On the other hand, the cross product captures the area of the parallelogram defined by a pair of vectors as well as how "perpendicular" two vectors are. It is maximal when the vectors are orthogonal and 0 when they

theorem (see section 5.9 in [2] for details); however, it is more difficult to express the complex version of the theorem using differential forms than it is using geometric algebra [3]. Moreover, while differential forms overlap with GC, they are not well-suited to the ideas encompassed by GA.

[3] "But I like commutativity better than associativity!" you say? I would argue that one of the most basic operations is function composition, which is associative, but not commutative. Furthermore, while it is possible to construct operations that are commutative, but not associative, they generally do not behave like multiplication.

are parallel. Both products also have a certain orientation to them. The dot product is nonnegative if the input vectors are parallel and nonpositive if they are antiparallel. The cross product flips sign when the multiplication order is reversed (i.e. it is anticommutative).

We now look in a bit more detail at how one might attempt to generalize the cross product. In n dimensions, the parallelogram defined by two vectors remains a 2D object, while the space orthogonal to these vectors becomes an $n-2$ dimensional object! It makes more sense to use the representation whose dimension is independent of the space it resides in. Notice also that regardless of whether we choose to use a 2D object or an $n-2$D object, we can no longer use vectors. Once we have suitably generalized to cross product to a 2D object, we might as well bring this generalization back to $\mathbb{R}^3$ and use a 2D object instead of vector, even though it may appear more complex. In the following sections we will see how this generalization can be done.

Exercise 1.1.7b in Folland (assigned as part of our very first problem set in 334) asked us to show that for any $a, b \in \mathbb{R}^3$, if there exists a $c \in \mathbb{R}^3 \setminus 0$ such that $a \cdot c = b \cdot c$ and $a \times c = b \times c$, then $a = b$. In other words, although the dot and cross products separately do not provide enough information to uniquely characterize a vector, taken together they provide enough information to recover equality. This hints at the idea that perhaps it is more enlightening to consider the dot and cross products simultaneously since there is a sense in which they preserve they information about a vector, but separate it into a parallel and a perpendicular piece.

Imagine, for a second, that we could add scalars and vectors just as we add real and imaginary numbers. What we would like to say is $a \cdot c + a \times c = b \cdot c + b \times c \iff a = b$. Let's massage this abuse of notation a little more: Let $a \star b = a \cdot b + a \times b$. Then $a \star c = b \star c \iff a = b$. This seems to imply that we are able to construct some notion of an inverse of c, allowing us to cancel it from both sides. The connection between the dot and cross products is made precise by GA.

2.2 Derivatives

We now turn our attention to vector derivatives. Unlike vector products, there are three commonly used derivatives: gradient, divergence, and curl. Unlike the dot and cross products, which seem to be somewhat reasonable products, these derivatives are perhaps even less intuitively the "right" derivatives to pick. Their definitions occasionally seem arbitrary, yet they are useful in a wide variety of contexts.

There is, at the very least, a notational correspondence between these derivatives and vector products since grad, div, curl are often denoted ∇, $\nabla\cdot$, and $\nabla\times$ respectively due to the relationships between their definitions and corresponding products. Unfortunately, in standard vector calculus, the gradient doesn't have a natural associated product while the other two do! This again hints that there is something missing from the conventional treatment of introductory analysis.

3 Geometric Algebra

Professor Macdonald first describes the properties of GA in n-dimensions, called $\mathbb{G}^n$. $\mathbb{G}^n$ is an extension of $\mathbb{R}^n$. It is a vector space that includes a new multiplication operation, known as the **geometric product**, $\mathbb{G}^n \times \mathbb{G}^n \mapsto \mathbb{G}^n$ that distributes over addition, is associative, has an identity, and commutes with a scalar. This product does not have an explicit symbol, but is instead used implicitly. In other words, $\forall a \in \mathbb{R}, \forall A, B, C \in \mathbb{G}^n$

$$\begin{aligned} A(B+C) = AB + AC, (B+C)A = BA + CA \\ (aA)B = A(aB) = a(AB) \\ (AB)C = A(BC) \\ 1A = A1 = A \end{aligned}$$

In particular, this product is not commutative. This should not come as a surprise however, since commutativity should not be expected in geometry, especially not in dimensions higher than 2. For example, pick an object around you and try composing two orthogonal 90-degree rotations.

Members of $\mathbb{G}^n$ are referred to as **multivectors**. There are two additional properties that link $\mathbb{R}^n$ and $\mathbb{G}^n$:

$$\mathbf{u}^2 = \mathbf{u}\mathbf{u} = \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \text{ for all } \mathbf{u} \in \mathbb{R}^n \tag{1}$$

Notice that this means every nonzero vector in $\mathbb{G}^n$ has an inverse $\mathbf{u}^{-1} = \dfrac{\mathbf{u}}{|\mathbf{u}|^2}$ s.t. $\mathbf{u}\mathbf{u^{-1}} = \mathbf{u^{-1}}\mathbf{u} = 1$. This is precisely the property we desired to have for the sum of the dot and cross products, and there is no corresponding property in conventional vector algebra. The formula for $\mathbf{u}^{-1}$ sometimes shows up in vector algebra formula (e.g. the projection and rejection formulas).

The most complicated property is that "every orthonormal basis for $\mathbb{R}^n$ determines a canonical basis... for the vector space $\mathbb{G}^n$." However, this will make sense in time.

Notice that by equation 1,

$$(\mathbf{u}+\mathbf{v})^2 = (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \mathbf{u}^2 + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v}^2$$

and

$$(\mathbf{u}+\mathbf{v})^2 = (\mathbf{u}+\mathbf{v})(\mathbf{u}+\mathbf{v}) = \mathbf{u}^2 + \mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} + \mathbf{v}^2$$

so

$$\mathbf{u} \cdot \mathbf{v} = \tfrac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}).$$

This equation means that the dot product captures the commutative, or symmetric, component of the geometric product.

Notice also that when $\mathbf{u}$ and $\mathbf{v}$ are orthogonal,

$$0 = \tfrac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}) \iff \mathbf{v}\mathbf{u} = -\mathbf{u}\mathbf{v} \tag{2}$$

3.1 Canonical Basis

Given an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$for $\mathbb{R}^n$, one can construct a **canonical basis** for $\mathbb{G}^n$ by taking all possible geometric products of basis vectors with indices in sorted order. This is sufficient since permuting the order of multiplication changes the product by a factor of ± 1 by (2). For example, given the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^3$, the canonical basis for $\mathbb{G}^3$ is

$$\begin{array}{rr}
1 & \text{basis for 0-vectors (scalars)} \\
\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3 & \text{basis for 1-vectors (vectors)} \\
\mathbf{e}_1\mathbf{e}_2 \quad \mathbf{e}_1\mathbf{e}_3 \quad \mathbf{e}_2\mathbf{e}_3 & \text{basis for 2-vectors (bivectors)} \\
\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 & \text{basis for 3-vectors (trivectors)}
\end{array}$$

There are a few things to observe about the canonical basis. Firstly $\dim \mathbb{G}^n = 2^n$ so it is a much more complicated space than $\mathbb{R}^n$, however this also makes for richer mathematics. Secondly, there is a correspondence between k-vectors (i.e. k-dimensional multivectors) and $n-k$-vectors in $\mathbb{G}^n$ since their subspaces have the same dimension. This property will be fleshed out in the duals section below. (Notice that the natural correspondence is to identify a k-dimensional basis vector with the $n-k$-dimensional vector that is the product of the other vectors. e.g. $\mathbf{e}_1$ and $\mathbf{e}_2\mathbf{e}_3$ should be duals of each other).

Now what do "bivectors", for example, represent? It is useful to break down a vector into 3 pieces: a magnitude, an orientation, and a sense. In particular, $\mathbf{a}$ and $-\mathbf{a}$ have the same magnitude and orientation, but opposite sense. We would consider these two vectors to be antiparallel. This concept can be generalized to higher dimensions. A bivector has a magnitude that corresponds to an area, an orientation that corresponds to the orientation of a plane in space, and a sense that corresponds to the direction of "circulation" in a plane. See the picture in the next section for more information. Notice this is similar to the cross product, which also has a magnitude and an orientation as well as a sense since there are two directions in which the cross product vector could point depending on the order in which two vectors are multiplied.

3.2 Geometric Algebra in 2D Subspaces

While it is possible to explicitly construct the geometric algebra (see [5]), it is fairly complicated to do in its full generality. We will thus restrict our attention for the time being to the geometric algebra in 2D subspaces on n-dimensional space. While this does not capture all the complexities of multivectors and the geometric product, it should hopefully give one a taste of how GA works.

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of a plane in $\mathbb{R}^n$ and two vectors $\mathbf{u} =$

$a\mathbf{e}_1 + b\mathbf{e}_2, \mathbf{v} = c\mathbf{e}_1 + d\mathbf{e}_2$ where $a, b, c, d \in \mathbb{R}$. Using equations 1 and 2,

$$\begin{aligned}
\mathbf{uv} &= (a\mathbf{e}_1 + b\mathbf{e}_2)(c\mathbf{e}_1 + d\mathbf{e}_2) \\
&= ac\mathbf{e}_1^2 + ad\mathbf{e}_1\mathbf{e}_2 + bc\mathbf{e}_2\mathbf{e}_1 + bd\mathbf{e}_2^2 \\
&= (ac + bd) + (ad - bc)\mathbf{e}_1\mathbf{e}_2 \qquad (3)
\end{aligned}$$

Notice the scalar part is precisely the dot product $\mathbf{u} \cdot \mathbf{v}$ while the second term has the same magnitude as the 2D cross product (evaluated by considering both vectors to have a third coordinate with value 0). However, it cannot be a vector since its square magnitude $(\mathbf{e}_1\mathbf{e}_2)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = -1 < 0$.

We call $(ad-bc)\mathbf{e}_1\mathbf{e}_2$ the **outer/wedge product** and write it as $\mathbf{u} \wedge \mathbf{v}$. This product has many interesting properties, which can be found in [4]. "Just as $\mathbf{u}$ represents an oriented length, $\mathbf{u} \wedge \mathbf{v}$ represents an oriented area." [3] Notice that the wedge product easily generalizes to n-dimensions, since it is straightforward to define the subspace spanned by two vectors $\mathbf{u}$ and $\mathbf{v}$ as well as the area and orientation of the parallelogram they define. The following figure from [4] is a visual representation of the wedge product.

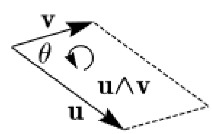

Figure 1: The outer product $\mathbf{u} \wedge \mathbf{v}$.

The orientation is commonly denoted by a circulation whose direction is given by moving from the direction the first vector is pointing to the direction the second vector is pointing. It is also important to note that while this object has an orientation, sense, and magnitude (area), it does not have a definite shape (i.e. it is not a parallelogram even though it is often convenient to draw or visualize it as such).

3.2.1 Fundamental Identity

With definitions in hand, equation 3 can be rewritten as

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$$

which is known as the fundamental identity. This formula connects our three products. Notice that the geometric product of two vectors is the sum of a scalar and a bivector. We are already seeing both the added complexity and the unification that GA provides. Notice also that this formula is not restricted to $\mathbb{G}^2$. We need only to choose an orthonormal basis for the subspace spanned by $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{G}^n$ to write the equality above.

There is another way to write the equality above:

$$\mathbf{uv} = \tfrac{1}{2}(\mathbf{uv} + \mathbf{vu}) + \tfrac{1}{2}(\mathbf{uv} - \mathbf{vu})$$

Since the first expression on the right is equal to $\mathbf{u} \cdot \mathbf{v}$, we have

$$\mathbf{u} \wedge \mathbf{v} = \tfrac{1}{2}(\mathbf{uv} - \mathbf{vu})$$

The expression on the right is anticommutative so the wedge product captures the anticommutative, or antisymmetric, part of the geometric product.

3.3 Complex Numbers

Geometric algebra contains the complex numbers as a subalgebra. In fact, we can define the complex numbers on any 2D subspace we wish! The consequences of this embedding are taken further in [4] than we do here. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be an orthonormal basis for a 2D subspace in $\mathbb{R}^n$. We define $\mathbf{i} = \mathbf{e}_1\mathbf{e}_2$ and note that this forms a basis for the bivectors of our subspace (this can be seen by explicitly constructing the canonical basis). $\mathbf{i}$ behaves exactly as i does in the complex numbers. In particular,

$$\mathbf{i}^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -\mathbf{e}_1^2\mathbf{e}_2^2 = -1$$

We now look at all objects of the form $a+b\mathbf{i}$ for $a, b \in \mathbb{R}$. Adding these objects clearly agrees with adding complex numbers. (Notice that the two components remain separate in both systems.) What about multiplication? Consider $a + b\mathbf{i}$ and $c + d\mathbf{i}$ for $a, b, c, d \in \mathbb{R}$. Then

$$(a + b\mathbf{i})(c + d\mathbf{i}) = ac + ad\mathbf{i} + bc\mathbf{i} + bd\mathbf{i}^2 = (ac - bd) + (ad + bc)\mathbf{i}$$

which agrees with complex multiplication.

We now look at some general results in $\mathbb{G}^n$.

3.4 Pseudoscalar

We can extend the idea of $\mathbf{i}$ as a basis for the bivectors in a 2D subspace to arbitrary dimension. If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ then we define $\mathbf{I} = \mathbf{e}_1\mathbf{e}_2 \cdots \mathbf{e}_n$ in $\mathbb{G}^n$as the **pseudoscalar** since the only multivectors of dimension n in $\mathbb{G}^n$ are scalar multiples of $\mathbf{I}$ and thus it is a one-dimensional subspace. Notice that $\mathbf{I}^{-1} = \mathbf{e}_n\mathbf{e}_{n-1} \cdots \mathbf{e_1}$ and that by swapping $\mathbf{e}_i$'s in the product, this must be $\pm\mathbf{I}$ depending on the dimension of the space we are considering.[4] In particular, $\mathbf{i}^{-1} = -\mathbf{i}$.

[4]For the remainder of the paper we will usually not be precise about which sign is the correct one to use since it often doesn't matter; however, we can always determine the correct sign if needed. See [4] for details.

3.5 Dual

The **dual** of a multivector A is

$$A^* = A\mathbf{I}^{-1}$$

Notice also that

$$A^{**} = A\mathbf{I}^{-1}\mathbf{I}^{-1} = \pm A\mathbf{I}\mathbf{I}^{-1} = \pm A$$

depending on the dimension of A.

3.6 Blades

A k-blade $\mathbf{B}$ is a special kind of multivector defined to be the product $\mathbf{b}_1\mathbf{b}_2\cdots\mathbf{b}_k$ of k nonzero orthogonal vectors or equivalently as the wedge product $\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k$ of k linearly independent vectors. This equivalence is proved in [4]. Blades are useful because they have many of the same properties as vectors and are thus written in boldface unlike general multivectors.

Define the **k-volume** as $|B| = |\mathbf{b}_1||\mathbf{b}_2|\cdots|\mathbf{b}_k|$. Then every blade $\mathbf{B}$ has an inverse $\mathbf{B}^{-1} = \pm\frac{\mathbf{B}}{|\mathbf{B}|^2}$ where the sign depends on the dimension of B. Other connections are discuss in the Projections and Rejections section and in [4].

3.7 Orthogonal Complement

If $\mathbf{A}$ is a j-blade, then $\mathbf{A}^*$ is an $(n-j)$-blade representing the orthogonal complement of $\mathbf{A}$. This notion is crucial for understanding the role of the dual and the pseudoscalar.

3.8 Cross-Wedge Duality

$$(\mathbf{u} \wedge \mathbf{v})^* = \mathbf{u} \times \mathbf{v}$$

since both are orthogonal to $\mathbf{u} \wedge \mathbf{v}$ and both have the same magnitude. We can also state this equality as

$$\mathbf{u} \wedge \mathbf{v} = (\mathbf{u} \times \mathbf{v})\mathbf{i}$$

where we have simply right-multiplied both sides by $\mathbf{i}$. Another useful form is

$$-\mathbf{u} \wedge \mathbf{v} = (\mathbf{u} \wedge \mathbf{v})\mathbf{i}^{-1}\mathbf{i}^{-1} = (\mathbf{u} \wedge \mathbf{v})^{**} = (\mathbf{u} \times \mathbf{v})^*$$

This relationship makes it clear why the wedge product is more readily generalized to arbitrary dimensions. In $\mathbb{G}^n$ the wedge product of two vectors is remains 2D, however the cross product must be represented by an $n-2$D object. In $\mathbb{R}^3$ this allows us to use the wedge product by proxy without using 2-dimensional objects, however in the process we lose the geometric connection to oriented areas that the wedge product provides.

3.9 Dot-Wedge Duality

$$(A \cdot B)^* = A \wedge B^*$$
$$(A \wedge B)^* = A \cdot B^*$$

for any two multivectors A, B.

3.10 Projections and Rejections

The **extended fundamental identity** is the fundamental identity for a vector and a k-blade. In other words, for any vector $\mathbf{a}$ and k-blade $\mathbf{B}$,

$$\mathbf{aB} = \mathbf{a} \cdot \mathbf{B} + \mathbf{a} \wedge \mathbf{B}$$

Note that any blade $\mathbf{B}$ in $\mathbb{G}^n$ represents the subspace spanned by the vectors that multiply together to become $\mathbf{B}$. Thus for any blade $\mathbf{B}$, we can decompose a vector $\mathbf{a} \in \mathbb{G}^n$ into the component parallel to $\mathbf{B}$ and the component orthogonal to $\mathbf{B}$. To make this precise, we can write $\mathbf{a} = \mathbf{a}_\parallel + \mathbf{a}_\perp$ where $\mathbf{a}_\parallel \in \mathbf{B}$ and $\mathbf{a}_\perp \perp \mathbf{B}$ so $\mathbf{a}_\parallel \wedge \mathbf{B} = \mathbf{0}$ and $\mathbf{a}_\perp \cdot \mathbf{B} = \mathbf{0}$. Thus by distributivity,

$$\mathbf{a} \cdot \mathbf{B} = \mathbf{a}_\parallel \mathbf{B} \iff \mathbf{a}_\parallel = (\mathbf{a} \cdot \mathbf{B})\mathbf{B}^{-1}$$
$$\mathbf{a} \wedge \mathbf{B} = \mathbf{a}_\perp \mathbf{B} \iff \mathbf{a}_\perp = (\mathbf{a} \wedge \mathbf{B})\mathbf{B}^{-1}$$

So the dot product can be thought of as encoding the parallel component (projection) of a vector w.r.t. some basis and the wedge product as encoding the orthogonal component (rejection) w.r.t some basis.

If $\mathbf{B}$ is a vector $\mathbf{b}$, then $(\mathbf{a} \cdot \mathbf{b})\mathbf{b}^{-1} = (\mathbf{a} \cdot \mathbf{b})\frac{\mathbf{b}}{|\mathbf{b}|^2}$, which is the usual vector projection formula.

$$(\mathbf{a} \wedge \mathbf{b})\mathbf{b}^{-1} = (\mathbf{ab} - \mathbf{a} \cdot \mathbf{b})\mathbf{b}^{-1} = \mathbf{abb}^{-1} - (\mathbf{a} \cdot \mathbf{b})\frac{\mathbf{b}}{|\mathbf{b}|^2} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\frac{\mathbf{b}}{|\mathbf{b}|^2}$$

is the usual vector rejection formula (although the GA formulation is more concise). Thus the formulas above generalize this decomposition in a very nice, coordinate-free way. If one wishes, he can impose a coordinate system after a formula has been derived.

4 Geometric Calculus

We now turn our attention to multivariable calculus using the GA toolkit we have developed. In geometric calculus, the gradient is defined as $\boldsymbol{\nabla} = \sum_j \mathbf{e}_j \partial_j$ where ∂_j is the partial derivative in the direction of $\mathbf{e}_j$. This is identical to the standard vector calculus definition; however, we can now manipulate it with the tools of geometric calculus. As in standard real analysis, we can treat $\boldsymbol{\nabla}$ as though it were a vector and compute the gradient of a scalar field $\boldsymbol{\nabla} f$, the divergence of a vector field $\boldsymbol{\nabla} \cdot \mathbf{f}$, and the curl of a vector field $\boldsymbol{\nabla} \times \mathbf{f}$.

Unlike vector calculus, we can make sense of $\boldsymbol{\nabla}\mathbf{f}$ where $\mathbf{f}$ is a vector field. As a corollary to the fundamental identity,

$$\boldsymbol{\nabla}\mathbf{f} = \boldsymbol{\nabla} \cdot \mathbf{f} + \boldsymbol{\nabla} \wedge \mathbf{f}$$

Notice that $(\boldsymbol{\nabla} \wedge \mathbf{f})^* = \boldsymbol{\nabla} \times \mathbf{f}$ so this formula connects the divergence, curl, and gradient.

Furthermore, the laplacian $\boldsymbol{\nabla}^2\mathbf{f} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{f})$ makes sense in GC, whereas in vector calculus one must write $\boldsymbol{\nabla}^2\mathbf{f} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{f}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f})$, which only works in $\mathbb{R}^3$. The equivalence of these two forms is not proved in the paper, so we will produce it here.

Lemma. *In* $\mathbb{R}^3$, $\boldsymbol{\nabla}^2\mathbf{f} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{f}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f})$

Proof:

$$\begin{aligned}
\boldsymbol{\nabla}^2\mathbf{f} &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{f}) \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{f} + \boldsymbol{\nabla} \wedge \mathbf{f}) && \text{(fundamental identity)} \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{f}) + \boldsymbol{\nabla}(\boldsymbol{\nabla} \wedge \mathbf{f}) && \text{(distributivity over addition)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{\nabla} \wedge \mathbf{f}) &= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{f}) + \boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{f}) && \text{(fundamental identity)} \\
&= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{f}) && (\boldsymbol{\nabla} \wedge \boldsymbol{\nabla} = 0) \\
&= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{f})^*\mathbf{i} && \text{(cross-wedge duality)} \\
&= (\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \times \mathbf{f}))^*\mathbf{i} && \text{(dot-wedge duality)} \\
&= -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) && \text{(cross-wedge duality)}
\end{aligned}$$

Notice that at no point in the proof did we have to invoke an explicit coordinate system.

4.1 Analyticity

Although we only looked at complex analyticity in 33X, geometric calculus provides a way of defining analytic functions in any dimension.

Looking at the complex case, let $f(x, y) = u(x, y) + v(x, y)\mathbf{i}$ where $\mathbf{i} = \mathbf{e}_1\mathbf{e}_2$. Then

$$\boldsymbol{\nabla} f = \mathbf{e}_1(u_x + v_x\mathbf{i}) + \mathbf{e_2}(u_y + v_y\mathbf{i}) = \mathbf{e}_1(u_x - v_y) + \mathbf{e}_2(v_x + u_y)$$

By the Cauchy-Riemann equations, $\boldsymbol{\nabla} f = 0 \iff f$ is analytic. In fact we can take $\boldsymbol{\nabla} f = 0$ to mean analytic and generalize this to n dimensions.[5]

According to Wikipedia, "A **manifold** is a... space that resembles Euclidean space near each point." This simplistic description is suitable for our purposes. Let M be a compact oriented m-dimensional manifold with boundary in $\mathbb{R}^n$ and

[5] In the 1-dimensional case, the only analytic functions are the constant functions, which differs from the conventional notion of real analytic functions.

$F(\mathbf{x})$ be a multivector valued field on M. Then we can define the **directed line integral** as

$$\int_M d^m\mathbf{x}F.$$

$d^m\mathbf{x} = \mathbf{I}_m(\mathbf{x})d^m x$ where $d^m x$ is an element of m-volume of M at $\mathbf{x}$ and $\mathbf{I}_m(\mathbf{x})$ is the pseudoscalar of the tangent space to M at $\mathbf{x}$. The order of the factors in the integrand is important since the geometric product is not commutative. It turns out that contour integrals are a special case of this integral.

We can consider the vector derivative on a manifold M as the projection of $\boldsymbol{\nabla}$ onto M, and we call this projection $\boldsymbol{\partial}$. With these definitions in hand, we can state

4.2 The Fundamental Theorem of (Geometric) Calculus

$$\int_M d^m\mathbf{x}\boldsymbol{\partial}F = \int_{\partial M} d^{m-1}\mathbf{x}F$$

This compact theorem encapsulates the many fundamental theorems of calculus we have developed in 33X. Due to the fundamental identity for derivatives, we can split up this integral into the divergence and curl while preserving the equality.

We can extend analyticity to manifolds as well by taking $\boldsymbol{\partial}F \equiv 0$. Cauchy's theorem on manifolds follows immediately since the left hand side of the equation becomes 0. Many other formulas from 336 can be written using GC. See [4] for details.

5 Conclusion

Geometric algebra and geometric calculus provide unified frameworks for talking about linear algebra and multivariable calculus in n-dimensions. Geometric algebra contains many important subalgebras (like the complex numbers and quaternions) and these subalgebras can be defined on any subspace with suitable dimension. Geometric algebra connects the dot and cross products, the three vector derivatives, and the fundamental theorems of calculus within a larger, more unified notational and conceptual framework.

Though these tools are considerably more sophisticated than those traditionally used in undergraduate analysis, they provide profound connections within the subject and it is worthwhile to invest time learning this machinery. There are a plethora of other applications of GA and GC that, for sake of time and scope, cannot be mentioned here. Firstly, one can rewrite all of linear algebra in GA while avoiding the need for coordinate systems as much as possible. This often leads to more concise proofs and strong geometric intuition. There are also many other subalgebras embedded in GA, in particular the quaternions and Pauli spin matrices. Several extensions of GA exist that are useful for working with projective spaces and encapsulating notions of intersection between objects of differing dimension, and these have found their way into computer graphics algorithms. GA is also useful for formulating physics results in insightful new ways. Rewriting Maxwell's equations using GC is particularly unifying and insightful, but one can also reformulate relativity and quantum mechanics using GC. For more information on these applications, see [4].

I hope this sampling of geometric algebra and geometric calculus inspires you to learn more about it and related algebras. They are powerful unified languages for multidimensional geometry.

References

[1] Artin, Emil. *Geometric Algebra*. Dover, 1957.

[2] Folland, G.B. *Advanced Calculus*.

[3] Hestenes, David. "A Unified Language for Mathematics and Physics". In: *Clifford Algebras and Their Applications in Mathematical Physics*. Ed. by Chisholm, J. S. R. and Common, A. K. Dordrecht: Springer Netherlands, 1986, pp. 1–23. ISBN: 978-94-009-4728-3. DOI: 10.1007/978-94-009-4728-3_1. URL: http://dx.doi.org/10.1007/978-94-009-4728-3_1.

[4] Macdonald, Alan. "A Survey of Geometric Algebra and Geometric Calculus". In: *Advances in Applied Clifford Algebras* 27.1 (2017), pp. 853–891. ISSN: 1661-4909. DOI: 10.1007/s00006-016-0665-y. URL: http://dx.doi.org/10.1007/s00006-016-0665-y.

[5] Macdonald, Alan. "An elementary construction of the geometric algebra". 2016. URL: http://faculty.luther.edu/~macdonal/GAConstruct.pdf.