Gap theorem

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Theorem 1. *Let p_k be a sequence of positive integers and let m be a positive integer such that $mp_{k+1} > (m+1)p_k$. Suppose the series $f(z) = \sum a_{p_k} z^{p_k}$ has radius of convergence 1. Then there is no point on the circle of convergence where f can be extended to be analytic.*

Proof. We can suppose that f can be extended to be analytic at 1. Let D be the unit disk. There is an open set W which contains 1 and D on which f is analytic. Let $h(w) = \frac{w^m + w^{m+1}}{2}$. There is an open set D_δ of radius $1 + \delta$ on which h is analytic and which maps into W. So $g(w) = f(h(w))$ has a power series expansion $g(w) = \sum c_k w^k$. If $z = h(w)$ with $|z| < 1$, then $g(w) = \sum a_{p_k}(h(w))^{p_k} = \sum c_k w^k$. This is true for an open set of w. Hence the two power series must be identical. So the coefficients are the same. Let $s_k(z) = \sum_0^k a_n z^n$. So it is a formal algebraic fact that the coefficients of $\sum a_{p_k}(h(w))^{p_k}$ and $\sum c_k w^k$ are the same. This is better than saying the values are the same – the formally algebraically computed coefficients are the same. Now if we can prove that $s_k(z)$ converges to a limit for some $|z| > 1$ we will have a contradiction since this will imply that the radius of convergence of f is larger than 1. But it is sufficient to prove that $s_{p_k}(z)$ converges since the gaps stabilize the sum s_k between the p_k's . ($s_n = s_{p_k}$ when $p_k \le n < p_{k+1}$.)

$$s_{p_k}(z) = s_{p_k}(h(w)) = \sum_{n=0}^{(m+1)p_k} c_n w^n, \; z = h(w).$$

This is formally true. There is nothing coming later that can contribute to the sum. The right hand side has a limit since $\sum c_k w^k$ converges. And this is true for some w such that $|z| > 1$.

Comment: Is $\sum_{n=0}^{p_k} a_n (h(w))^n$ a partial sum of $\sum c_n w^n$? Yes, when there are gaps it is $\sum_{n=0}^{(m+1)p_k} c_n w^n$, and since the latter converges even when $|z| = |h(w)| > 1$, we have a contradiction. □