Cauchy-Riemann equations

March 31, 2011

Definition 1. *Let $f(z)$ be a complex valued function defined in a neighborhood of a point $z_1 = x_1 + iy_1$. Then f is* ***complex differentiable*** *at z_1 if*

$$f(z) = f(z_1) + p(z)(z - z_1), \tag{1}$$

where p is continuous at z_1. We define the complex derivative of f at z_1 to be the value of p at z_1.

$$f'(z_1) = p(z_1). \tag{2}$$

Theorem 1. *Let $f(z) = u(x, y) + iv(x, y)$. Then f is complex differentiable at z_1 if and only if u and v are real differentiable at (x_1, y_1) and*

$$u_x(x_1, y_1) = v_y(x_1, y_1),\ u_y(x_1, y_1) = -v_x(x_1, y_1). \tag{3}$$

These are called the Cauchy-Riemann equations.

Proof. Let us write equation (1) in matrix form:

$$\begin{bmatrix} u(x,y) \\ v(x,y \end{bmatrix} = \begin{bmatrix} u(x_1,y_1) \\ v(x_1,y_1) \end{bmatrix} + \begin{bmatrix} a(x,y) & -b(x,y) \\ b(x,y) & a(x,y) \end{bmatrix} \begin{bmatrix} x - x_1 \\ y - y_1 \end{bmatrix} \tag{4}$$

We can write this line by line

$$u(x,y) = u(x_1,y_1) + a(x,y)(x - x_1) - b(x,y)(y - y_1) \tag{5}$$
$$v(x,y) = v(x_1,y_1) + b(x,y)(x - x_1) + a(x,y)(y - y_1). \tag{6}$$

In this form we see that u and v are real differentiable at (x_1, y_1) and $u_x = v_y,\ u_y = -v_x$ at (x_1, y_1).

Next suppose u and v are real differentiable at (x_1, y_1) and (3) is true. Let $a_1 = u_x(x_1, y_1) = v_y(x_1, y_1),\ b_1 = v_x(x_1, y_1) = -u_y(x_1, y_1)$. Let's write the definition of real differentiability as follows

$$u(x,y) = u(x_1,y_1) + a_1(x - x_1) - b_1(y - y_1) + e(x,y) \tag{7}$$
$$v(x,y) = v(x_1,y_1) + b_1(x - x_1) + a_1(y - y_1) + d(x,y), \tag{8}$$

where $e(x,y)/|z - z_1| \to 0, d(x,y)/|z - z_1| \to 0$ as $z \to z_1$. So

$$f(z) = f(z_1) + (a_1 + ib_1)(z - z_1) + \epsilon(z)(z - z_1),\ \text{where } \epsilon(z) = \frac{e(z) + id(z)}{z - z_1}. \tag{9}$$

and $\epsilon(z) \to 0$ as $z \to z_1$. This can be rewritten as

$$f(z) = f(z_1) + p(z)(z - z_1), \tag{10}$$

where $p(z) = a_1 + ib_1 + \epsilon(z)$ and $p(z) \to a_1 + ib_1$ as $z \to z_1$ so f is complex differentiable at z_1. □