

The Twelve-Tone Method of Composition

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Abstract

Twelve-tone composition requires the non-repeating use of every note of the twelve-tone octave. The rules governing twelve-tone composition provide groundwork for the application of group theory and other mathematics. This paper provides several examples illustrating the use of group theory to describe musical patterns in twelve-tone composition. It is based on the paper *Mathematics and the Twelve-Tone System: Past, Present, and Future* by Robert Morris [2].

Contents

1	Introduction	2
2	A Klein Four-Group	3
3	Identities	5
4	Theorems Involving T- and I-Matrices	6
5	Discussion and Conclusion	8
6	Works Cited	9

1 Introduction

Mathematics and the Twelve-Tone System: Past, Present, and Future by Robert Morris [2] takes a survey of the applications of group theory to twelve-tone composition. It begins chronologically and continues by building on previous results. It starts by explaining the most commonly known real-world example of a Klein Four-Group, and then goes on to discuss identities involving the elements of that group. It then discusses some theorems that can be generalized to non-twelve-tone composition, and then discusses matrices that represent the systems. Then, it summarizes the current research and proposes future directions for the field.

In modern music theory, there are twelve notes in one octave. Twelve-tone composition is a special way of arranging these twelve notes. Arnold Schoenberg invented twelve-tone composition in 1908 to break free of what “the ear had gradually become acquainted with” [1]. Twelve-tone composition requires the use of each of the twelve notes without repetition until each note has been played once. After each note has been played once, the next twelve notes must follow the same rules. This continues until the end of the piece.

The application of mathematics to twelve-tone composition depends on the definition of a row, which is a specific permutation of the twelve tones. It can be represented by an array of twelve notes. Each note is represented by a number between 0 and 11. For example, the chromatic scale is represented by:

[0 1 2 3 4 5 6 7 8 9 10 11]

The row can also be represented with written music¹:



Figure 1: Chromatic Scale

As seen in the array and the scale, 0 in the array represents a C, a 1 represents a D flat, etc. 11, the last note of the octave, represents a B.²

A unique row is a unique permutation of the twelve notes. We begin with transformations on a row.

¹All written music in this paper was generated using software found at noteflight.com.

²This paper assumes that the reader understands the basic written representation of musical notes. If the reader is unfamiliar with written music, <http://readsheetmusic.info/readingmusic.shtml> is recommended.

2 A Klein Four-Group

The most basic (and earliest) example of the application of mathematics to the twelve-tone method of composition involves the following basic transformations that can be performed on a unique twelve-tone row. In the illustrations, the top staff is an original row, and the bottom staff is the row after the transformation has been performed.

The retrograde transformation reverses the order of the sequence of notes in the row.



Figure 2: Retrograde (R)

In the example above, the original row is:

[0 1 6 2 7 9 3 4 10 11 5 8]

The transformed row is:

[8 5 11 10 4 3 9 7 2 6 1 0]

The inversion transformation turns each note "upside down." It takes the interval between each note and the base note (C or 0), and flips the interval in the opposite direction. In the array, this is done by subtracting each element from 12 (and using modulus 12, so the base note remains in the same octave). The resulting sequence maintains the same intervals between consecutive notes, in the opposite direction.



Figure 3: Inversion (I)

In the example above, the original row is the same as the one in the previous example. The transformed row is:

[0 11 6 10 5 3 9 8 2 1 7 4]

The retrograde-inversion transformation is a combination of the retrograde and inversion transformations. Note: This can be done by performing the R operation followed by the I operation, or by performing the I operation followed by the R operation.



Figure 4: Retrograde-Inversion (RI)

In this example, the original row is again the same as the one in the previous examples. The transformed row is:

$$[4\ 7\ 1\ 2\ 8\ 9\ 3\ 5\ 10\ 6\ 11\ 0]$$

The identity transformation maintains the original row.



Figure 5: Identity (P)

Following is the Cayley table for a Klein Four-Group comprised by the four transformations.

	P	R	I	RI
P	P	R	I	RI
R	R	P	RI	I
I	I	RI	P	R
RI	RI	I	R	P

This is a commonly used real-world example of a Klein Four-Group (a Klein Four-Group is a cyclic group of order 2). Note that these operations are commutative.

While the representation of these four operations as a Klein Four-Group shows what Schoenberg saw as “the absolute and unitary perception of musical space” [1], there are many other possible operations to take into account, such as transposition.

3 Identities

Not only do rows of twelve-tone composition fit in Klein Four-Groups, but they have some other interesting properties that can be represented as mathematical identities. Morris mentions some identities involving the already defined operations as well as some more (defined below). We will briefly discuss some of them. Here are some necessary definitions:

- An array P represents a row of twelve notes.
- P_k is the k th element (note) of this row.
- $i(P_a, P_b) = P_b - P_a$ is the interval between P_a and P_b . In music theory, the interval between two notes is the “distance” between those two notes. In the notation used here, it is the difference between the numerical representations of the two notes.
- $T_n P_k$ is a transposition of P_k by n notes. In music theory, transposition is increasing a note by a certain interval. In the notation used here, we add the number n (representing the size of the interval to transpose by) to the note P_k .
- The array $\text{INT}(P)$ represents the intervals of P ; $\text{INT}(P) = [i(P_0, P_1), i(P_1, P_2), \dots, i(P_{10}, P_{11})]$

Following are some identities involving these definitions. The identities to which the reader wishes to pay attention is up to the reader’s discretion; only few need to be understood for the following sections.

- $i(P_a, P_b) = -i(P_b, P_a)$
This can be seen from the given definition of an interval. The interval between a note and the next is the opposite of the interval between the next note and the first.
- $i(T_n P_a, T_n P_b) = i(P_a, P_b)$
Translation of two notes by the same interval does not affect the interval between those two notes.
- $i(I P_a, I P_b) = i(P_b, P_a)$
This can be seen from the definition of Inversion (subtracting the note from 12) and the first identity in this list.
- $\text{INT}(T_n P) = \text{INT}(P)$
Translation does not affect the intervals between consecutive notes. This can be seen by applying the second identity eleven times to the row.
- $\text{INT}(T_n I P) = I(\text{INT}(P))$
We see from the previous identity that the T_n can be ignored. Then, we see that the succession of intervals in the inverse is the same as the inverse of the succession of intervals.

- $\text{INT}(\text{RT}_n\text{P}) = \text{R}(\text{INT}(\text{IP}))$
Again, the T_n can be ignored. The succession of intervals after the Retrograde operation is the same as the reverse of the succession of intervals after the Inverse operation.
- $\text{INT}(\text{RIT}_n\text{P}) = \text{R}(\text{INT}(\text{P}))$
The succession of intervals after the RI operation has been performed is the same as the reverse of the succession of intervals after the Inversion has been performed.
- $i(\text{P}_a, \text{T}_n\text{P}_a) = n$
The interval between a note and its transposition is equal to the distance the note was transposed by. This can be seen from the given definition of transposition.
- $\text{P}_a + \text{T}_n\text{IP}_a = n$
This can be seen from the given definitions: $a + [n + (12 - a)] = n + 12$. Since we are doing arithmetic modulus twelve, this equals n .

To illustrate the use of these transformations in real music, one can see that Schoenberg's Violin Concerto (Op. 36) uses the twelve-tone technique. The basic row used is P:

$$[0 \ 1 \ 6 \ 2 \ 7 \ 9 \ 3 \ 4 \ 10 \ 11 \ 5 \ 8]$$

Throughout the piece, Schoenberg uses several of the operations mentioned to aesthetically complement this basic row. For example:

RT_5IP :

$$[9 \ 0 \ 6 \ 7 \ 1 \ 2 \ 8 \ 10 \ 3 \ 11 \ 4 \ 5]$$

T_{11}P :

$$[11 \ 0 \ 5 \ 1 \ 6 \ 8 \ 2 \ 3 \ 9 \ 10 \ 4 \ 7]$$

It is impressive that all of the previously mentioned material was studied before mathematics was officially introduced into twelve-tone composition. Mathematics was first introduced through pitch-classes and transpositional levels (involving the physics of sound waves), then with group theory (involving sets, permutations, etc.), and then with the modern mathematical reasoning found in music theory (for example, the proof of the Complement Theorem (mentioned in section 4)). We continue with the group theory portion.

4 Theorems Involving T- and I-Matrices

Note:

- $\#X$ is the cardinality of (number of items in) the set X .

- $\text{mul}(A, B, n)$ is the multiplicity (number of times it appears in the set) of $\{i(a, b) = n\}$ for all $a \in A$ and $b \in B$.
- $\text{sum}(A, B, n)$ is the number of sums $a + b = n$, where $a \in A$ and $b \in B$.
- A' is the complement of A .

Following are three theorems developed in the late 20th century.

Transpositional Common Tone Theorem: $\#(A \cap T_n B) = \text{mul}(A, B, n)$

Inversional Common Tone Theorem: $\#(A \cap T_n IB) = \text{sum}(A, B, n)$

Complement Theorem: $\text{mul}(A', B', n) = 12 - (\#A + \#B) + \text{mul}(A, B, n)$

In 1974, Bo Alphonse (Yale) invented invariance matrices. A T-matrix E is defined by: $E_{i,j} = X_i + IY_j$. An I-matrix F is defined by: $F_{i,j} = X_i + Y_j$. Following are the T- and I- matrices for a trichord (three notes following a special set of rules) and a hexachord (six notes). We are using $X = [0\ 1\ 2\ 4\ 7\ 8]$, $Y = [3\ 4\ 8]$ for the example.

Table 1: T-matrix						
	0	1	2	4	7	8
9	9	10	11	1	4	5
8	8	8	10	0	3	4
4	4	5	6	8	10	0

Table 2: I-matrix						
	0	1	2	4	7	8
3	3	4	5	7	10	11
4	4	5	6	8	11	0
8	8	9	10	0	3	4

The T- and I- matrices can also be made for a twelve-tone row P (let $X = Y = P$).

Let us define the function $\text{ifunc}_n(X, Y)$ as the number of instances of n in the T-matrix. Then, $\text{ifunc}_n(X, IY)$ is the number of instances of n in the I-matrix. Here are some resulting identities:

- $\text{ifunc}_n(X, Y) = \text{mul}(X, Y, n)$
The number of times that the interval between two notes (one from each set of notes) is n is equal to the number of times n appears in the T-matrix. The intervals between notes from each set are used to choose which notes are played at the same time for aesthetic purposes.
- $\text{ifunc}_n(X, IY) = \text{sum}(X, Y, n)$
The number of times two notes (one from each set) add up to n is equal to the number of times n appears in the I-matrix.

Following are the theorems that come from the definitions above:

Corollary of Transpositional Common Tone Theorem: $\#(X \cap T_n Y) = \text{ifunc}_n(X, Y)$

Corollary of Inversional Common Tone Theorem: $\#(X \cap T_n IY) = \text{ifunc}_n(X, IY)$

Let G be the T-matrix: $G_{i,j} = X_i + IX_j$. Let us use $X = [0\ 10\ 7\ 9\ 2\ 8]$, which is the first hexachord of Stravinsky's "A Sermon, a Narrative, and a Prayer."

0	10	7	9	2	8
2	0	9	11	4	10
5	3	0	2	7	1
3	1	10	0	5	11
10	8	5	7	0	6
4	2	11	1	6	0

Rotate G so that the diagonals (top-left to bottom-right) become columns. Continue with the diagonal six rows down to have six elements per column in the rotated matrix.

0	10	7	9	2	8
0	9	11	4	10	2
0	2	7	1	5	3
0	5	11	3	1	10
0	6	11	8	5	7
0	4	2	11	1	6

Columns 0 and 3, like twelve-tone rows, are I-invariant, meaning that they follow the previously mentioned identities for twelve-tone rows involving the operation I (section 4). Columns 1 and 5 are Inversions of each other, and columns 2 and 4 are Inversions of each other.

5 Discussion and Conclusion

Current research on the application of group theory to twelve-tone composition involves combinatorial arrays (studying the given arrays through the lens of combinatorics), transformations other than those mentioned, and pitch classes. A pitch class is a set of notes that are all octaves apart. Since the physics of sound waves is applicable to the study of pitch classes, current research is combining the two.

At this point, the original paper states that all of the applications of group theory to twelve-tone composition have already been studied, and proposes other (related) directions for future research. For example, modern music theory is

based on the definition of twelve notes per octave; what if we define a different number? Yes, many of the given identities and theorems still apply, but pitch-classes and tonality will be affected; we still need the distance between notes to divide the length of an octave.

Perhaps another direction of research could be to study the effect of the defined transformations on the actual sound. Modern music theory and physics have provided reasons for why what “the ear had gradually become acquainted with” [1] sounds pleasant, and modern composers have found ways to make even twelve-tone compositions fit both sets of rules (pleasant yet twelve-tone) by adding multiple simultaneous parts to a song, each following its own series of twelve-tone rows. Maybe in order to take twelve-tone composition into account, we need to re-study the aesthetics of sound.

6 Works Cited

1. Schoenberg, Arnold. “Composition with Twelve Tones.” *Style and Idea: Selected Writings*. Univ. California Press, Berkeley, 1975. Web. May 2015. <http://www.toddtarantino.com/hum/compositionwithtwelvetones.html>.
2. Morris, Robert. “Mathematics and the Twelve-Tone System: Past, Present, and Future.” *Mathematics and Computation in Music*. 2007: 266-288.