Existence of Linear Hypercyclic Operators on Infinite-Dimensional Banach Spaces

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1 Introduction

The notion of hypercyclicity was first introduced by operator theorists studying the connections between cyclic vectors and the invariant subspace problem, which, for Hilbert spaces (Banach spaces where the norm is induced from an inner product), remains an open problem (Enflo constructed a Banach space and a linear operator that does not a have nontrivial invariant subspace [7]). The invariant subspace problem asks if every bounded linear operator on a space possesses a nontrivial, closed invariant subspace. A vector is cyclic with respect to a bounded linear operator if the span of its orbit is dense in the containing space. Hypercyclic vectors are a special case of cyclic vectors and are related to a similar problem called the invariant subset problem, although, this isn't discussed in this paper

In 1969, S. Rolewicz posed the problem "Does an infinite-dimensional, separable Banach space support a hypercyclic operator?" [6]. It was proven later that the answer is the affirmative. In this paper, we examine and summarize a proof of this existence result, which was given by L. Bernal-González [1]. We begin by setting preliminary definitions and terminology [1][4]. Then, we provide some related intermediate results and then present the existence theorem. A proof is given using those intermediate results. Afterwards, we present some results on hypercyclicity in a related context along with open problems and discuss applications of hypercyclicity to dynamics and chaos.

2 Preliminaries

We begin this paper by laying out the necessary preliminary definitions that will be needed.

2.1 Metric Spaces and Completeness

Definition 2.1. Let X be a nonempty set. A *metric* (or *distance function*) is a function $d : X \times X \to \mathbb{R}$ that satisfies the following properties:

- $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) \geq 0$ for all $x, y \in X$ (non-negativity)
- $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry)
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (Triangle Inequality)

The pair (X, d) is called a *metric space*. When there is no ambiguity, X will be used instead to denote a metric space.

Definition 2.2. Let X be a metric space. A sequence of points $\{x_n\}$ in X *converges* to a point $x \in X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > N$.

Definition 2.3. Let X be a metric space. A sequence of points $\{x_n\}$ in X is *Cauchy* if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \in N$.

Remark. Convergent sequences are Cauchy. Choose $N \in \mathbb{N}$ large such that $d(x_n, x) < \epsilon/2$ when $n > N$. Then, $d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \epsilon/2 + \epsilon/2 = \epsilon$ for all $m, n > N$.

Definition 2.4. A metric space is said to be *complete* if every Cauchy sequence of points $\{x_n\}$ in X converges to a point $x \in X$.

2.2 Linear Spaces and Norms

Definition 2.5. A *linear space* (or *vector space*) V over a scalar field $\mathbb{K}$ (for our purposes, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) is a set of points (or vectors) on which element addition and scalar multiplication are defined. They must satisfy the addition axioms given above as well as the following.

1. If $x, y \in V$ and $\lambda, \mu \in \mathbb{K}$, then $\lambda x + \mu y \in V$ (closure under vector addition and scalar multiplication)

2. If $x, y \in V$ and $\lambda, \mu \in \mathbb{K}$, then $\lambda(\mu x) = (\lambda\mu)x$, $\lambda(x+y) = \lambda x + \lambda y$ and $(\lambda+\mu)x = \lambda x + \mu x$.

Definition 2.6. If V is a linear space, then a *seminorm* on V is a function $f : V \to \mathbb{R}$ with following properties:

- $f(x) \geq 0$ for all $x \in V$ (non-negativity)
- If $x \in V$ and $\lambda \in \mathbb{K}$, then $f(\lambda x) = |\lambda| f(x)$
- $f(x+y) \leq f(x) + f(y)$ for all $x, y \in V$ (Triangle Inequality)

If, in addition, $f(x) = 0$ if and only if $x = 0$, then f is called a *norm* and is more commonly denoted by $||\cdot|| : V \to \mathbb{R}$. Intuitively, norms define a notion of vector "length" or "magnitude". In cases of possible confusion, we write $||x||_V$ instead of $||x||$ to denote the norm of $x \in V$ with respect to the norm defined on V.

Definition 2.7. A *normed linear space* is a pair $(V, ||\cdot||)$, where V is a linear space and $||\cdot||$ is a norm. Normed linear spaces are metric spaces with respect to the metric $d(x, y) = ||x - y||$.

Definition 2.8. Let X be a linear space and $S = \{v_1, \ldots, v_n\} \subset X$. A vector w is a *linear combination* of the vectors $v_1, \ldots, v_n$ if there exist scalars $c_1, \ldots, c_n \in \mathbb{K}$ such that

$$w = \sum_{k=1}^{n} c_k v_k$$

Definition 2.9. Let X be a linear space and $S \subset X$. We define the *span* of S to be

$$\text{span } S = \{w : w = c_1 v_1 + \cdots + c_k v_k \text{ for } v_1, \ldots, v_k \in S \text{ and } c_1, \ldots, c_k \in \mathbb{K}\}$$

Note that in the above definition, the linear combinations are taken over a finite set of vectors, even if S is infinite.

2.3 Separability

Definition 2.10. Let X be a metric space and E be a subset of X. A point $x \in X$ is a limit point of E if there exists a sequence of points $\{x_n\}$ in E that converge to x.

Definition 2.11. Let E be a subset of a metric space X. E is *dense* in X if every point of X is a limit point of E. Equivalently, E is dense in X if $\overline{E} = X$ since $\overline{E}$ is the union of E and all of its limit points.

Definition 2.12. A metric space X is *separable* if there exists a dense subset $E \subset X$ that is countable.

2.4 Banach Spaces, Linear Operators and Dual Spaces

Definition 2.13. A *Banach space* is a complete, normed linear space.

Definition 2.14. Let X and Y be Banach spaces over a scalar field $\mathbb{K}$. $T : X \to Y$ is a *linear operator* if

- $T(x+y) = T(x) + T(y)$ for all $x, y \in X$
- $T(\lambda x) = \lambda T(x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$

We often write Tx instead of $T(x)$. It is clear that compositions of linear operators are linear. Indeed, if $S : X \to Y$ and $T : Y \to Z$ are two linear operators, where X, Y and Z are Banach spaces, then

$$(S \circ T)(x+y) = S(T(x+y)) = S(T(x) + T(y)) = S(T(x)) + S(T(y)) = (S \circ T)(x) + (S \circ T)(y)$$

and

$$(S \circ T)(\lambda x) = S(T(\lambda x)) = S(\lambda T(x)) = \lambda S(T(x)) = \lambda (S \circ T)(x)$$

so that the composition $S \circ T$ of S and T is also a linear operator.

Definition 2.15. Let X and Y be two Banach spaces. A linear operator $T : X \to Y$ is *bounded* if there exists positive $C \in \mathbb{R}$ such that $||Tx|| \leq C||x||$ for all $x \in X$. Otherwise, T is *unbounded*. We denote the set of linear operators by $\mathcal{L}(X,Y)$ and the set of bounded linear operators by $\mathcal{B}(X,Y)$. If $X = Y$, we simply write $\mathcal{L}(X)$ and $\mathcal{B}(X)$, respectively.

In general, it isn't obvious that all linear operators are bounded. For finite-dimensional spaces, this is the case. However, there are many examples of unbounded linear operators in infinite-dimensional spaces; the boundedness also depends on the norms used.

As an example, let $C^1([0,1])$ be the set of continuously differentiable functions $[0,1]$ and $C^0([0,1])$ be the set of continuous functions on $[0,1]$. If both are equipped with the sup-norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$, then the differentiation operator $D : C^1([0,1]) \to C^0([0,1])$ is unbounded. To see this, note that the set of $p_n(x) = x^n$ $(n \geq 1)$ is a subset of $C^1[0,1]$. Furthermore, $||p_n|| = 1$ for all n. However,

$$||Dp_n||_\infty = ||np_{n-1}||_\infty = n||p_{n-1}||_\infty = n$$

which is unbounded. If instead, we equip $C^1([0,1])$ with the C^1 norm, given by

$$||f||_{C^1} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$$

(and $C^0([0,1])$ with the sup-norm), then $D : C^1([0,1]) \to C^0([0,1])$ is bounded. To see this,

$$||Df||_\infty = \sup_{x\in[0,1]} |f'(x)| \leq \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)| = ||f||_{C^1}$$

So, $||D|| = 1$ in this case.

Definition 2.16. If $T : X \to Y$ is a bounded linear operator, then we define the *operator norm* (or *uniform norm*) $||T||$ of T by

$$||T|| = \inf\{M : \forall x \in X, ||Tx|| \leq M||x||\}$$

The operator norm defines a norm on $\mathcal{B}(X,Y)$, which can be verified as follows. Since X and Y have norms defined on them so that $||x|| \geq 0$ and $||y|| \geq 0$ for all $x \in X$ and $y \in Y$, it is clear that $\{M : \forall x \in X, ||Tx|| \leq M||x||\}$ consists only of nonnegative real numbers. So, the operator norm is nonnegative. Furthermore, $||T|| = 0$ if and only if T maps all $x \in X$ to $0 \in Y$. If $\lambda \in \mathbb{K}$, then, $||\lambda Tx|| = \lambda||Tx||$ so that

$$||\lambda T|| = \inf\{M : \forall x \in X, ||\lambda Tx|| \leq M||x||\} = \inf\left\{M : \forall x \in X, ||Tx|| \leq \frac{M}{|\lambda|}||x||\right\} = |\lambda| \cdot ||T||$$

Let $A = \{M : \forall x \in X, ||(S+T)(x)|| \leq M||x||\}$, where $S, T \in \mathcal{B}(X,Y)$ and $(S+T)(x) = Sx + Tx$. Then,

$$||(S+T)(x)|| = ||Sx + Tx|| \leq ||Sx|| + ||Tx|| \leq ||S|| \cdot ||x|| + ||T|| \cdot ||x|| = (||S|| + ||T||) \cdot ||x||$$

So, $||S|| + ||T|| \in A$. Since $||S+T|| = \inf A$, $||S+T|| \leq ||S|| + ||T||$.

An immediate and useful consequence of this definition is that $||Tx|| \leq ||T|| \cdot ||x||$ for all $x \in X$. T is bounded if $||T||$ finite. The converse is also true. If T is bounded, $S = \{M : \forall x \in X, ||Tx|| \leq M||x||\}$ is nonempty and is bounded below. By the greatest-lower-bound property of the reals, $\inf S$ exists and is finite. So, $||T||$ exists and is finite.

Theorem 2.1. *If $S : X \to Y$ and $T : Y \to Z$ are two bounded linear operators, where X, Y and Z are Banach spaces, then $||ST|| \leq ||S|| \cdot ||T||$.*

Proof. Let $A = \{M : \forall x \in X, ||(S \circ T)x|| \leq M||x||\}$. Fix $x \in X$. Then,

$$||(S \circ T)x|| = ||S(Tx)|| \leq ||S|| \cdot ||Tx|| \leq ||S|| \cdot ||T|| \cdot ||x||$$

This shows that $||S|| \cdot ||T|| \in A$. The desired inequality follows immediately since $||ST||$ is a lower bound of A by definition. □

Corollary 2.1.1. *If $T \in \mathcal{B}(X)$, then $||T^n|| \leq ||T||^n$, for any $n \in \mathbb{N}_0$.*

Definition 2.17. Let X be a Banach space. A *linear functional* (or *linear form*) on X is a linear operator $T : X \to \mathbb{R}$. The set of linear functionals $\mathcal{L}(X, \mathbb{R})$ is called the *algebraic dual space* of X and the set of bounded linear functionals $\mathcal{B}(X, \mathbb{R})$ is called the *topological dual space* of X. For our purposes, we'll denote the topological dual space of X by X^* and shorten the term to "dual space".

Theorem 2.2. *Let $T : X \to Y$ be a linear operator. Then, T is bounded if and only if T is continuous.*

Proof. Suppose T is bounded. Then, $||T|| = M$ is well-defined and finite. Given $\epsilon > 0$, choosing $\delta = \epsilon/M$ shows us

$$||Tx - Ty|| = ||T(x - y)|| \leq ||T|| \cdot ||x - y|| < M \cdot \frac{\epsilon}{M} = \epsilon$$

so that T is continuous.

Suppose T is continuous. Then, T is continuous at $0 \in X$. $T(0) = 0$ by linearity. Take $\epsilon = 1$ in the continuity definition. Then, there exists $\delta > 0$ such that if $||x|| < \delta$, then $||Tx|| < \epsilon = 1$. For any $x \neq 0$, define $\tilde{x} = \alpha\delta x/||x||$ (any $0 < \alpha < 1$ will do). Then, $||\tilde{x}|| = |\alpha\delta| \cdot ||x||/||x|| = \alpha\delta < \delta$ so that $||T\tilde{x}|| < 1$. By linearity,

$$||Tx|| = \left|\left|T\left(\frac{||x||\tilde{x}}{\alpha\delta}\right)\right|\right| = \frac{||x||}{\alpha\delta}||T\tilde{x}|| < \frac{1}{\alpha\delta}||x||$$

Since $x \in X$ was chosen arbitrarily, $\{M : \forall x \in X, ||Tx|| \leq M||x||\}$ is nonempty and has an infimum. Hence, T is bounded. □

Continuing the example following **Definition 2.15**, we see that the derivative operator on the set of continuously differentiable functions on the interval $[0, 1]$ equipped with the sup-norm is discontinuous.

2.5 Orbits and Hypercyclicity

Definition 2.18. Let X be a Banach space and $T : X \to X$ be a bounded linear operator. We define the *orbit* of a vector $x \in X$ with respect to T by

$$\text{Orb}(T, x) = \{T^n x : n \in \mathbb{N}_0\}$$

where the notation T^n denotes a composition of T with itself n times. Note that $\text{Orb}(T, x)$ is an "at most countable" set. Orbits can be defined similarly for nonlinear and unbounded operators as well.

Definition 2.19. Let X be a separable Banach space and $T : X \to X$ be a bounded linear operator. A vector $x \in X$ is *hypercyclic* for T if $\text{Orb}(T, x)$ is dense in X. A bounded linear operator T is *hypercyclic* if it has a hypercyclic vector.

Remark. The assumption of separability for the existence of a hypercyclic operator is necessary since if X isn't separable, then X has no countable dense subset and hence, $\text{Orb}(T, x)$ cannot be dense in X, regardless of the operator T and vector x chosen. Non-separable spaces do not have hypercyclic operators.

Interestingly, linear hypercyclicity also occurs only on infinite-dimensional spaces. It was shown by Rolewicz that no linear operator is hypercyclic on any finite-dimensional space [6].

3 The Existence Theorem and its Proof

3.1 Intermediate Results

We are now ready to discuss the results that lead up to the proof of the existence theorem. First, a couple useful lemmas are given. Next, two intermediate results (**Theorem 3.3** and **Theorem 3.4**) necessary for the proof of the main existence theorem are stated, the proofs of which, are not given. Instead, we refer to their original authors (see [3] and [2], respectively). For convenience, we define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 3.1. *Let X be a Banach space. Let $S : X \to X$ be a linear operator and I be the identity operator. Define $T = I + S$. Then, $T^n = \sum_{k=0}^{n} \binom{n}{k} S^k$, where $S^0 = I$.*

Proof. We proceed by induction on n. The base case is already proven by the definition of T since

$$T^1 = T = I + S = \binom{1}{0} S^0 + \binom{1}{1} S^1 = \sum_{k=0}^{1} \binom{1}{k} S^k$$

Suppose $T^n = \sum_{k=0}^{n} \binom{n}{k} S^k$ for some n. Then,

$$T^{n+1} = T(T^n) = T\left(\sum_{k=0}^{n} \binom{n}{k} S^k\right) = I\left(\sum_{k=0}^{n} \binom{n}{k} S^k\right) + S\left(\sum_{k=0}^{n} \binom{n}{k} S^k\right) = \sum_{k=0}^{n} \binom{n}{k} S^k + \sum_{k=0}^{n} \binom{n}{k} S^{k+1} =$$
$$\sum_{k=0}^{n} \binom{n}{k} S^k + \sum_{k=1}^{n+1} \binom{n}{k-1} S^k = S^0 + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] S^k + S^{n+1} =$$
$$\sum_{k=0}^{n+1} \binom{n+1}{k} S^k$$

We conclude that $T^n = \sum_{k=0}^{n} \binom{n}{k} S^k$ for all $n \in \mathbb{N}$ by induction. □

Lemma 3.2. *Let $\{x_k\}$ be a dense sequence in a separable metric space X, with metric d. Suppose $\{y_k\}$ is another sequence in X such that $\lim_{k\to\infty} d(y_k, x_k) = 0$. Then, $\{y_k\}$ is dense in X.*

Proof. Fix $w \in X$. Since $\{x_k\}$ is dense in X, w is a limit point of $\{x_k\}$. Let $N_r(w)$ denote an open ball (neighborhood) of radius $r > 0$, centered around w. If there exists some $\epsilon > 0$ such that $N_\epsilon(w)$ only contains finitely many points $x_{k_1}, \ldots, x_{k_n}$ of $\{x_k\}$, then choosing $\epsilon' < \min\{\epsilon, d(w, x_{k_1}), \ldots, d(w, x_{k_n})\}$, we see that $N_{\epsilon'}(w)$ contains no points of $\{x_k\}$. But then w isn't a limit point of $\{x_k\}$, a contradiction. So, every neighborhood of w contains infinitely many points of $\{x_k\}$.

Let $\epsilon > 0$ be given. Let $\{x_{k_j}\}$ be the subset (subsequence) of $\{x_k\}$ such that $\{x_{k_j}\} \subset N_{\epsilon/2}(w)$ $(j \in \mathbb{N})$. Since $\lim_{k\to\infty} d(y_k, x_k) = 0$, there exists J large such that if $j > J$, $d(y_{k_j} - x_{k_j}) < \epsilon/2$. Then, for all $j > J$,

$$d(y_{k_j} - w) \leq d(y_{k_j} - x_{k_j}) + d(x_{k_j} - w) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, we see that $N_{\epsilon/2}(w)$ contains infinitely points in $\{y_k\}$. Since ϵ was chosen arbitrarily, w is also a limit point of $\{y_k\}$. Since w was chosen arbitrarily in X, it follows that all points of X are limit points of $\{y_k\}$. Hence, $\{y_k\}$ is also dense in X. □

Theorem 3.3. *Let X be an infinite-dimensional, separable Banach space and let X^* be the dual space of X. Then, there exists sequences $\{e_n\}_0^\infty \subset X$ and $\{\varphi_n\}_0^\infty \subset X^*$ with the following properties:*

1. *$\varphi_n(e_m) = \delta_{mn}$ for all $m, n \in \mathbb{N}$, where $\delta_{mn} = 1$ if $m = n$ and 0 if $m \neq n$, is the Kronecker delta.*
2. *$\overline{span\{e_n : n \in \mathbb{N}_0\}} = X$*
3. *If $\varphi_n(x) = 0$ for all $n \in \mathbb{N}_0$, then $x = 0$.*
4. *$||e_n|| = 1$ for all $n \in \mathbb{N}_0$ and $\sup_{n\in\mathbb{N}_0} ||\varphi_n|| < \infty$.*

Theorem 3.4. *Let C_n be a $2^k \times 2^k$ matrix, whose entries are given by $C_{ij}(n) = \binom{n}{2^k+j-i}$. Let B_n be a column vector with 2^k entries, whose ith entry $b_i(n)$ is a polynomial in n such that $\deg b_i(n) \leq 2^k - i$. Then, for sufficiently large n, there exists a solution X_n to the equation $B_n = C_n X_n$. Furthermore, the entries of X_n satisfy $|x_i(n)| \leq C/n^i$, where C is a constant dependent on k.*

3.2 The Existence Theorem

Theorem 3.5. *Let X be an infinite-dimensional, separable Banach space. Then, there exists $T \in \mathcal{B}(X)$ such that T is hypercyclic.*

Proof. Let I be the identity operator $Ix = x$. Let $\{a_n\}_1^\infty$ be a sequence of positive real numbers such that $\sum_{k=1}^\infty a_n$ converges. Define $S : X \to X$ by

$$Sx = \sum_{k=0}^{\infty} a_{k+1}\varphi_{k+1}(x)e_k \tag{1}$$

where $\{e_n\}_0^\infty \subset X$ and $\{\varphi_n\}_0^\infty \subset X^*$ are furnished by **Theorem 3.3**. Since $\sup_{n\in\mathbb{N}_0} ||\varphi_n|| = C < \infty$ and $||e_n|| = 1$ (see **Property 4** from **Theorem 3.3**),

$$||S|| \le \sum_{k=0}^{\infty} ||a_{k+1}\varphi_{k+1}e_k|| \le \sum_{k=0}^{\infty} a_{k+1}||\varphi_{k+1}|| \cdot ||e_k|| \le C\sum_{k=0}^{\infty} a_{k+1} < \infty$$

where we take operator norms above. So, $S \in \mathcal{B}(X)$. Now, we claim that $T = I + S$ is a hypercyclic operator on X. To do this, we construct a hypercyclic vector $y \in X$ for T.

From **Property 2** from **Theorem 3.3**, since $\text{span}\{e_n : n \in \mathbb{N}_0\}$ is dense in X, we can choose a dense sequence $\{z_k\}_1^\infty \subset X$, where each z_k is given by a linear combination of the e_n's; $z_k = \sum_{i=0}^{2^k-1} z_{i,k}e_i$, for well-chosen scalars $z_{i,k}$. Our goal is to construct an increasing sequence of positive integers $\{n_j\}_1^\infty$ and vectors $y_j = \sum_{i=2^j}^{2^{j+1}-1} b_ie_i$ $(j \in \mathbb{N})$ such that

$$||y_j|| \le 2^{-j}(1 + ||T||)^{-n_{j-1}} \tag{2}$$

and

$$\left|\left|T^{n_j}\left(\sum_{k=1}^{j} y_k\right) - z_j\right|\right| \le 2^{-j} \tag{3}$$

for which we then show that $y = \sum_{k=1}^\infty y_k$ is hypercyclic for T. The construction of $\{n_j\}_1^\infty$ and $\{y_j\}_1^\infty$ follows inductively. For the sake of convenience, define $n_0 = 0$ and $b_0 = b_1 = 0$.

We begin by noting that by **Property 1** from **Theorem 3.3**, $Se_i = a_ie_{i-1}$ for $i \in \mathbb{N}$ and $Se_0 = 0$. We can see inductively that

$$S^m e_i = \begin{cases} \left(\prod_{k=0}^{m-1} a_{i-k}\right) e_{i-m}, m \le i \\ 0, m > i \end{cases} \tag{4}$$

To construct $y_1 = b_2e_2 + b_3e_3$ and n_1 in the base case $n = 1$, we must satisfy the following:

$$||y_1|| = ||b_2e_2 + b_3e_3|| \le 2^{-1}(1 + ||T||)^{-n_0} = \frac{1}{2}$$

and

$$||T^{n_1}y_1 - z_1|| = ||b_2T^{n_1}e_2 + b_3T^{n_1}e_3 - z_{0,1}e_0 - z_{1,1}e_1|| \le \frac{1}{2}$$

Expanding $T^{n_1}y_1$ using (4) and **Lemma 3.1**,

$$\begin{aligned} T^{n_1}y_1 &= b_2\left(e_2 + \binom{n_1}{1}a_2e_1 + \binom{n_1}{2}a_2a_1e_0\right) + b_3\left(e_3 + \binom{n_1}{1}a_3e_2 + \binom{n_1}{2}a_3a_2e_1 + \binom{n_1}{3}a_3a_2a_1e_0\right) \\ &= \left[\binom{n_1}{2}b_2a_2a_1 + \binom{n_1}{3}b_3a_3a_2a_1\right]e_0 + \left[n_1b_2a_2 + \binom{n_1}{2}b_3a_3a_2\right]e_1 + [b_2 + n_1b_3a_3]\,e_2 + b_3e_3 \end{aligned}$$

Then, we obtain the following for $T^{n_1}y_1 - z_1$, noting that $z_1 = z_{0,1}e_0 + z_{1,1}e_1$.

$$\left[\binom{n_1}{2}b_2a_2a_1 + \binom{n_1}{3}b_3a_3a_2a_1 - z_{0,1}\right]e_0 + \left[n_1b_2a_2 + \binom{n_1}{2}b_3a_3a_2 - z_{1,1}\right]e_1 + [b_2 + n_1b_3a_3]\,e_2 + b_3e_3 \quad (5)$$

Since we only require one solution triplet (b_2, b_3, n_1) that satisfies (2) and (3), we impose the following condition to simplify our search,

$$\varphi_\alpha(T^{n_1}y_1 - z_1) = 0 \text{ for } \alpha = 0, 1$$

so that $T^{n_1}y_1 - z_1 = (b_2 + n_1b_3a_3)e_2 + b_3e_3$. Since $\varphi_0(e_i) = 0$ for $i = 1, 2, 3$, $\varphi_1(e_i) = 0$ for $i = 0, 2, 3$, and $\varphi_0(e_0) = 1 = \varphi_1(e_1) = 1$ (see **Property 1** from **Theorem 3.3**), the equations $\varphi_0(T^{n_1}y_1 - z_1) = 0$ and $\varphi_1(T^{n_1}y_1 - z_1) = 0$ give the following linear system.

$$\binom{n_1}{2}b_2a_2a_1 + \binom{n_1}{3}b_3a_3a_2a_1 = z_{0,1} \implies \binom{n_1}{2}b_2 + \binom{n_1}{3}b_3a_3 = \frac{z_{0,1}}{a_2a_1}$$

$$n_1b_2a_2 + \binom{n_1}{2}b_3a_3a_2 = z_{1,1} \implies n_1b_2 + \binom{n_1}{2}b_3a_3 = \frac{z_{1,1}}{a_2}$$

We can rewrite this system as follows.

$$C_nX_n = \begin{bmatrix} \binom{n_1}{2} & \binom{n_1}{3} \\ \binom{n_1}{1} & \binom{n_1}{2} \end{bmatrix}\begin{bmatrix} b_2 \\ b_3a_3 \end{bmatrix} = \begin{bmatrix} z_{0,1}/a_2a_1 \\ z_{1,1}/a_2 \end{bmatrix} = B_n$$

Having written the system in this form, we can apply **Theorem 3.4** since the entries of B_n are polynomials in n_1 of degree 0 (which is less than $2^k - i = 2 - i$, where i is the row index of a component in B_n). For n_1 sufficiently large, there exists a constant P such that the above system has a solution $[b_2, b_3a_3]$ and $|b_2| \le P/n_1$ and $|b_3a_3| \le P/n_1^2$. We choose n_1 large so that b_2 and b_3 are determined,

$$||b_2e_2 + b_3e_3|| \le ||b_2|| + ||b_3|| \le \left|\frac{P}{n_1}\right| + \left|\frac{P}{a_3n_1^2}\right| \le \frac{1}{2}$$

and

$$||T^{n_1}y_1 - z_1|| = ||(b_2 + n_1b_3a_3)e_2 + b_3e_3|| \le ||b_2|| + ||n_1b_3a_3|| + ||b_3|| \le \left|\frac{P}{n_1}\right| + \left|\frac{P}{n_1}\right| + \left|\frac{P}{a_3n_1^2}\right| \le \frac{1}{2}$$

Thus, we have found y_1 and n_1 that satisfy (2) and (3) and the base case is established.

Now, suppose $n_0 < \cdots < n_{k-1}$ and $y_1, \ldots, y_{k-1}$ have been found such that they satisfy (2) and (3). We wish to find $y_k = \sum_{j=2^k}^{2^{k+1}-1} b_je_j$ and n_k that satisfy (2) and (3). Again, we impose the following condition.

$$\varphi_p\left(T^{n_k}\left(\sum_{j=1}^{k} y_i\right) - z_k\right) = 0 \text{ for } p = 0, 1, \ldots, 2^k - 1 \quad (6)$$

Expanding the left side by applying linearity of φ_p and T^{n_k}, and **Lemma 3.1**,

$$\begin{aligned}
\varphi_p\left(T^{n_k}\left(\sum_{j=1}^{k} y_i\right) - z_k\right) &= \sum_{i=1}^{k} \varphi_p(T^{n_k}y_i) - \varphi_p(z_k) \\
&= \sum_{i=1}^{k}\sum_{j=2^i}^{2^{i+1}-1} b_j\varphi_p(T^{n_k}e_j) - \varphi_p\left(\sum_{i=0}^{2^k-1} z_{i,k}e_i\right) \\
&= \sum_{j=2}^{2^{k+1}-1} b_j\varphi_p(T^{n_k}e_j) - \sum_{i=0}^{2^k-1} z_{i,k}\varphi_p(e_i) \\
&= \sum_{j=2}^{2^{k+1}-1} b_j\varphi_p\left[\sum_{r=0}^{n_k}\binom{n_k}{r}S^re_j\right] - z_{p,k}
\end{aligned}$$

From here, we apply (4) (for $j \geq r$ since all terms where $j < r$ are zero by (4)). Furthermore, by **Property 1** from **Theorem 3.3**, $\varphi_p(e_{j-r}) = 0$ for $j - r \neq p$ and $\varphi_p(e_{j-r}) = 1$ for $j - r$. So, $r = j - p$. Then,

$$\begin{aligned}\sum_{j=2}^{2^{k+1}-1} b_j \varphi_p \left[\sum_{r=0}^{n_k} \binom{n_k}{r} S^r e_j \right] - z_{p,k} &= \sum_{j=2}^{2^{k+1}-1} \sum_{r=0}^{n_k} b_j \binom{n_k}{r} \left[\prod_{q=j-r+1}^{j} a_q \right] \varphi_p(e_{j-r}) - z_{p,k} \\ &= \sum_{j=2}^{2^{k+1}-1} b_j \binom{n_k}{j-p} \left[\prod_{q=p+1}^{j} a_q \right] - z_{p,k} \\ &= 0\end{aligned}$$

Separating the unknown variables b_j ($j = 2^k, \ldots, 2^{k+1} - 1$) from the known values b_j ($j = 2, \ldots, 2^k - 1$),

$$\sum_{j=2^k}^{2^{k+1}-1} b_j \binom{n_k}{j-p} \left[\prod_{q=p+1}^{j} a_q \right] = z_{p,k} - \sum_{j=2}^{2^k-1} b_j \binom{n_k}{j-p} \left[\prod_{q=p+1}^{j} a_q \right] \implies$$

$$\left[\prod_{q=p+1}^{2^k} a_q \right] \sum_{j=2^k}^{2^{k+1}-1} b_j \binom{n_k}{j-p} \left[\prod_{q=2^k+1}^{j} a_q \right] = z_{p,k} - \sum_{j=2}^{2^k-1} b_j \binom{n_k}{j-p} \left[\prod_{q=p+1}^{j} a_q \right] \implies$$

$$\sum_{j=2^k}^{2^{k+1}-1} b_j \binom{n_k}{j-p} \left[\prod_{q=2^k+1}^{j} a_q \right] = \frac{1}{\left[\prod_{q=p+1}^{2^k} a_q \right]} \left(z_{p,k} - \sum_{j=2}^{2^k-1} b_j \binom{n_k}{j-p} \left[\prod_{q=p+1}^{j} a_q \right] \right)$$

Denote the entire right-hand side of the above equation by α_p. Since $p = 0, 1, \ldots, 2^k - 1$, we have a linear system of 2^k equations, which can be expressed in the following way.

$$C_n X_n = \begin{bmatrix} \binom{n_k}{2^k} & \cdots & \binom{n_k}{2^{k+1}-1} \\ \vdots & \ddots & \vdots \\ \binom{n_k}{1} & \cdots & \binom{n_k}{2^k} \end{bmatrix} \begin{bmatrix} b_{2^k} \\ a_{2^k+1} b_{2^k+1} \\ \vdots \\ \left(\prod_{j=2^k+1}^{2^{k+1}-2} a_j \right) b_{2^{k+1}-2} \\ \left(\prod_{j=2^k+1}^{2^{k+1}-1} a_j \right) b_{2^{k+1}-1} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{2^k-1} \end{bmatrix} = B_n$$

Note that

$$\binom{n_k}{j-p} = \frac{n_k!}{(n_k - j + p)!(j-p)!} = \frac{1}{(j-p)!} \left(\frac{n_k!}{(n_k - j + p)!} \right) = CP(n_k)$$

where C is a constant and $P(n_k)$ is a polynomial in n_k of degree $j - p$. If i indexes the components of B_n, then $\alpha_p = \alpha_{i-1}$. Furthermore, since $\max\{j\} = 2^k - 1$,

$$\max\{j - p\} = (2^k - 1) - (i - 1) = 2^k - i$$

Hence, each α_{i-1} is a polynomial in n_k of degree at most $2^k - i$. Applying **Theorem 3.4** is now justified. For n_k sufficiently large (and $n_k \geq n_{k-1}$), $C_n X_n = B_n$ has a solution. Furthermore, there exists a constant P such that $\left| \left(\prod_{j=2^k+1}^{2^k+i} a_j \right) b_{2^k+i} \right| \leq P/n_k^i$, where $i = 0, 1, \ldots, 2^k - 1$. Using these facts, we now find n_k and

b_j $(i = 2^k, \ldots, 2^{k+1}-1)$ such that (2) and (3) are satisfied. First, we expand $T^{n_k}\left(\sum_{j=1}^k y_j\right) - z_k$.

$$\begin{aligned}
T^{n_k}\left(\sum_{j=1}^{k} y_j\right) - z_k &= \sum_{j=1}^{k} T^{n_k} y_j - \sum_{j=0}^{2^k-1} z_{j,k} e_j \\
&= \sum_{j=1}^{k-1} \sum_{i=2^j}^{2^{j+1}-1} b_i T^{n_k} e_i + \sum_{j=2^k}^{2^{k+1}-1} b_j T^{n_k} e_j - \sum_{j=0}^{2^k-1} z_{j,k} e_j \\
&= \sum_{j=2}^{2^k-1} b_j \left(\sum_{i=0}^{n_k} \binom{n_k}{i} S^i e_j\right) + \sum_{j=2^k}^{2^{k+1}-1} b_j \left(\sum_{i=0}^{n_k} \binom{n_k}{i} S^i e_j\right) - \sum_{j=0}^{2^k-1} z_{j,k} e_j \\
&= \sum_{j=2}^{2^k-1} \sum_{i=0}^{n_k} b_j \binom{n_k}{i} S^i e_j - \sum_{j=0}^{2^k-1} z_{j,k} e_j + \sum_{j=2^k}^{2^{k+1}-1} \sum_{i=j+1}^{n_k} b_j \binom{n_k}{i} S^i e_j \\
&+ \sum_{j=2^k}^{2^{k+1}-1} \sum_{i=0}^{j} b_j \binom{n_k}{i} S^i e_j
\end{aligned}$$

Consider the first two sums. Since $j = 0, \ldots, 2^k - 1$, using our requirement from (6), we see that they drop out. Furthermore, using (4), the third summation term drops out since $i > j$. So, we see that

$$T^{n_k}\left(\sum_{j=1}^{k} y_j\right) - z_k = \sum_{j=2^k}^{2^{k+1}-1} \sum_{i=0}^{j} b_j \binom{n_k}{i} S^i e_j$$

From here, we apply **Property 4** from **Theorem 3.3** and **Corollary 2.1.1** to obtain

$$\begin{aligned}
\left\|T^{n_k}\left(\sum_{j=1}^{k} y_j\right) - z_k\right\| &= \left\|\sum_{j=2^k}^{2^{k+1}-1} \sum_{i=0}^{j} b_j \binom{n_k}{i} S^i e_j\right\| \\
&\leq \sum_{j=2^k}^{2^{k+1}-1} \sum_{i=0}^{j} |b_j| \binom{n_k}{i} ||S||^i \cdot ||e_j|| \\
&= \sum_{j=2^k}^{2^{k+1}-1} \left(\sum_{i=0}^{j} \binom{n_k}{i} ||S||^i\right) |b_j|
\end{aligned}$$

Recall that $||S|| \leq \sup_{n \in \mathbb{N}_0} ||\varphi_n|| \sum a_k < \infty$ so that $||S||^i$ is well defined. Furthermore, as stated earlier, as a consequence of **Theorem 3.4**, each $|b_j| \leq P/n_k^i$ $(j = 2^k + i - 1$, where i is the component index of b_j in $B_n)$, for some constant P. Hence, the sum of all $|b_j|$ is bounded above by Q/n_k, for some constant Q (it is clear that the bound is of order n_k^{-1} since $|b_{2^k}| \leq P/n_k$ has the highest order). So,

$$\left\|T^{n_k}\left(\sum_{j=1}^{k} y_j\right) - z_k\right\| \leq \frac{Q}{n_k}$$

Now, choose n_k sufficiently large, with $n_k > \max\{2^k Q, n_{k-1}\}$, so that the b_j's are all determined and $|b_j| \leq 4^{-k}(1 + ||T||)^{-n_{k-1}}$ $(j = 2^k, \ldots, 2^{k+1} - 1)$. We require $n_k > n_{k-1}$ so that $\{n_k\}$ forms an increasing sequence of natural numbers. We require $n_k > 2^k Q$ so that

$$\left\|T^{n_k}\left(\sum_{j=1}^{k} y_j\right) - z_k\right\| \leq \frac{Q}{n_k} < \frac{Q}{2^k Q} = 2^{-k}$$

which is condition (3). Using **Property 4** from **Theorem 3.3**, the third condition on n_k that gives the bound for $|b_j|$ gives

$$||y_k|| = \left|\left|\sum_{j=2^k}^{2^{k+1}-1} b_j e_j\right|\right| \leq \sum_{j=2^k}^{2^{k+1}-1} |b_j| \cdot ||e_j|| = \sum_{j=1}^{2^k} |b_{2^k+j-1}|$$
$$\leq 4^{-k}(1+||T||)^{-n_{k-1}} \left(\sum_{j=1}^{2^k} 1\right) = 2^{-k}(1+||T||)^{-n_{k-1}}$$

This is condition (2). Thus, y_k satisfies (2) and (3). The induction is complete and so is the construction of $\{y_k\}_1^\infty$ and $\{n_k\}_1^\infty$.

Now, take $y = \sum_{k=1}^\infty y_k$. To show that y is a hypercyclic vector for T, it suffices to show that $\text{Orb}(T, y)$ has a dense subset by showing $\lim_{k\to\infty} ||T^{n_k}y - z_k|| = 0$, using the fact that $\{z_k\}$ is dense in X and **Lemma 3.2**. Then, note the following.

$$||T^{n_k}y - z_k|| = \left|\left|T^{n_k}\left(\sum_{j=1}^\infty y_j\right) - z_k\right|\right| = \left|\left|T^{n_k}\left(\sum_{j=1}^k y_j\right) - z_k + T^{n_k}\left(\sum_{j=k+1}^\infty y_j\right)\right|\right|$$
$$\leq \left|\left|T^{n_k}\left(\sum_{j=1}^k y_j\right) - z_k\right|\right| + \left|\left|T^{n_k}\left(\sum_{j=k+1}^\infty y_j\right)\right|\right|$$
$$\leq 2^{-k} + \left|\left|\sum_{j=k+1}^\infty T^{n_k} y_j\right|\right| \leq 2^{-k} + \sum_{j=k+1}^\infty ||T^{n_k}|| \cdot ||y_j||$$
$$\leq 2^{-k} + \sum_{j=k+1}^\infty 2^{-j}(1+||T||)^{n_k}(1+||T||)^{-n_{j-1}}$$
$$\leq 2^{-k} + \sum_{j=k+1}^\infty 2^{-j} = 2^{-k} + 2^{-k}\sum_{j=1}^\infty 2^{-j} = 2^{-k} + 2^{-k} = 2^{-k+1}$$

Sending $k \to \infty$, $2^{-k+1} \to 0$ so that $||T^{n_k}y - z_k|| \to 0$. So, $\text{Orb}(T, y)$ is dense in X. Hence, y is a hypercyclic vector for T and T is a hypercyclic operator on X. □

4 Hypercyclicity in a Different Context

4.1 Hypercyclicity in F-Spaces

A more general framework under which hypercyclicity has been more extensively studied is that of the F-space. Loosely speaking, a F-space is a topological vector space (vector space endowed with some topology) that is metrizable, that is, its topology is compatible with some metric d so that the open sets in the topology are unions of open balls $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$; the metric is translation invariant and the space is complete with respect to d. Banach spaces are F-spaces since they are complete by definition and metrics induced by norms are translation invariant.

$$d(x, y) = ||x - y|| = ||(x + z) - (y + z)|| = d(x + z, y + z)$$

In the 1980s, a useful characterization of hypercyclicity was found, independently by Kitai [11] and Gethner and Shapiro [12], called the Hypercyclicity Criterion. It provides a set of sufficient conditions for a continuous linear operator to be hypercyclic. A slightly weaker form of the criterion due to Bès and Peris [14] is given below.

Theorem 4.1. *Let X be a separable F-space and T be a continuous linear operator on X. Suppose there are dense subsets $X_0 \subset X$ and $Y_0 \subset X$, an increasing sequence of positive integers $\{n_k\}$ and a sequence of mappings $S_{n_k} : Y_0 \to X$ such that as $k \to \infty$,*

1. *for every $x \in X_0$, $T^{n_k} x \to 0$*
2. *for every $y \in Y_0$, $S_{n_k} y \to 0$*
3. *for every $y \in Y_0$, $(T^{n_k} \circ S_{n_k})(y) \to y$*

Then, T is hypercyclic on X.

As an example to show the applications of this criterion, we (informally) show that the differentiation operator D defined on the set of analytic functions on the open unit disk $\mathcal{O}(\mathbb{D})$ (with a norm chosen so that D is bounded) is hypercyclic. D is linear and continuous. Let $\mathcal{P}(\mathbb{D}) \subset \mathcal{O}(\mathbb{D})$ be the set of polynomials defined on $\mathbb{D}$. Since analytic functions are given by a power series, every function $\mathcal{O}(\mathbb{D})$ can be approximated uniformly by a sequence of polynomials. In particular, $\mathcal{P}(\mathbb{D})$ is dense in $\mathcal{O}(\mathbb{D})$. Define $Sf = \int_0^z f(\zeta)\, d\zeta$, $z \in \mathbb{D}$. S is also linear. Since f is analytic and $\mathbb{D}$ is star-shaped, it has a complex primitive on $\mathbb{D}$ so S is a well-defined operator, sending $\mathcal{O}(\mathbb{D})$ to itself since the primitive is analytic on $\mathbb{D}$ by definition. Define $S_n = S^n$. Fix $p \in \mathcal{P}(\mathbb{D})$. $D^n p \to 0$ as $n \to \infty$ just by repeated differentiation (since exponents are finite). $S^n p \to 0$ as $n \to \infty$ as well since the coefficients of each term in $S^n p$ tend to zero as $n \to \infty$ and $|z|^{n+1} \leq |z|^n$ in $\mathbb{D}$. Finally, D and S are inverse operations by construction so $(T^n \circ S_n)(p) = p$ for all n. Hence, D satisfies the hypercyclicity criterion. So, D is hypercyclic.

MacLane showed that there exists an entire function f such that $\{f^{(n)}\}$ is dense in the set of entire functions $\mathcal{O}(\mathbb{C})$ [13]. In particular, the derivative operator D is hypercyclic on the space of entire functions.

4.2 Open Problems

Hypercyclicity is an area of active research. There is still much we don't understand. Below, we present some open problems mentioned in [9].

1. Does every hypercyclic operator on a Banach/Hilbert space satisfy the Hypercyclicity Criterion?
2. Does there exist a continuous linear operator T on a Hilbert space X such that every vector $x \in X$ is hypercyclic for T?
3. Do all separable, infinite-dimensional F-spaces support a hypercyclic operator?
4. What are the common characteristics of topological vector spaces that support a hypercyclic operator?

5 Dynamics and Chaos

Finally, we discuss some of the applications of hypercyclicity to dynamical systems and, in particular, chaos. Dynamical systems have tremendous applications as they provide tools to model various physical phenomena. They can describe the motion of various objects in Euclidean space, the flow of fluids, various engineering systems, etc.

5.1 Dynamics

Dynamics is the study of the evolution of the states of a system. The states are elements of some metric space X and the evolution is described by a continuous operator $T : X \to X$. In the discrete case, the progression of the states of the system are often indexed by some countable set. If $x_n \in X$ is the current system's state, then $x_{n+1} = Tx_n$ gives the system's next state. Usually, when X is clear, the dynamical system (X, T) is simply written as $T : X \to X$.

$T : X \to X$ is a linear dynamical system when T is a continuous linear map. Naturally, studying a given discrete linear system's set of states and its evolution after an arbitrary number of iterations given an initial state x_0 will involve studying the orbit of the initial state with respect to T; each element in $\text{Orb}(T, x_0)$ is some future state that will be achieved after a certain number of (time) steps. Intuitively, hypercyclic operators already model some rather interesting behavior since $\text{Orb}(T, x_0)$ is dense in the X. Any state in the state space can be approximated to arbitrary precision by some state in $\text{Orb}(T, x_0)$ given that one "waits long enough".

5.2 Chaos

A subtopic in linear dynamics is that of a chaotic system. Despite being deterministic, chaotic systems are characterized by rather erratic and unpredictable behavior and extreme sensitivity to the initial conditions. Small changes in the system often lead to wildly different behavior.

Chaotic systems are subject of great interest as there are many naturally occurring phenomena that exhibit chaotic behavior and have applications to other fields as well. Solutions of the Lorenz equations in modeling atmospheric conditions are chaotic. The logistic map, used for modeling population growth/decline, is chaotic. In physics, the motion of the double pendulum (a pendulum with another pendulum on its end) and the magnetic pendulum is chaotic. In computer science, pseudo-random number generation is derived from chaotic systems [8].

To define chaos, we first need a characterization of "sensitivity" to the initial conditions. For this, we have the property of sensitive-dependence on initial conditions.

Definition 5.1. Let (X, d) be a metric space and T be a continuous map of X to X. The dynamical system $T : X \to X$ has *sensitive-dependence on initial conditions* if there exists $\delta > 0$ such that for every $x \in X$ and $\epsilon > 0$, there exists $x' \in X$ such that $d(x, x') < \epsilon$ and $d(T^n x, T^n x') > \delta$, for some n. Such a δ is referred to as a sensitivity constant for T.

Intuitively, if a dynamical system is sensitive to its initial conditions, a slight change in the initial state can lead to states with significantly different characteristics after some number of time steps or evolutions. In view of this, the above definition simply says that there is some "distance between states" $\delta > 0$ (measuring how different states are) such that no matter how "close" the initial states x and x' are, their evolved states after applying T a certain number n times iteratively to x and x' result in states that are at least δ apart in distance.

A couple more definitions are required to fully characterize chaos.

Definition 5.2. Let $T : X \to X$ be a linear dynamical system. A point $x \in X$ is a *periodic point* of T if $T^n x = x$ for some n. x is a *fixed point* of T if $n = 1$ ($Tx = x$).

Definition 5.3. A dynamical system $T : X \to X$ is *topologically transitive* if for any pair of nonempty open sets $U \subset X$ and $V \subset X$, there exists $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$. $T : X \to X$ is called *mixing* if this occurs for all $n > N$, for some N.

With the above definitions, we are ready to state a commonly accepted definition of chaos that was proposed by Devaney in 1986.

Definition 5.4. A dynamical system $T : X \to X$ is *chaotic* if it has sensitive-dependence on initial conditions, T is topologically transitive and T has a dense set of periodic points.

Remark. It can be shown that if the metric space X has no isolated points, then, if T is topologically transitive and has a dense set of periodic points, then $T : X \to X$ has sensitive-dependence on initial conditions.

When chaos theory was first being developed, chaotic behavior was associated with nonlinearity as linear systems were believed to be "predictable" and "well-behaved". However, with the discovery of linear operators that have dense orbits, albeit on infinite-dimensional spaces (hypercyclicity), the study of chaos in a linear setting became a subject of significant interest. In the case of linear dynamical systems, topological transitivity is equivalent to hypercyclicity (see **Birkhoff Transitivity Theorem**). Furthermore, it can be shown that hypercyclicity by itself implies sensitive-dependence on initial conditions. By **Theorem 3.5** proven above, on any Banach space, we can find some linear operator that is hypercyclic and has sensitive-dependence on initial conditions. Such an operator already exhibits most characteristics of chaotic behavior.

6 Conclusion

We've seen that hypercyclicity plays a crucial role in the study of chaotic systems and linear dynamical systems in general. Chaotic operators have widespread applications as they can be used to model the behavior of various systems found in nature. Hypercyclicity is a subject of current research and there are still many open problems concerning it. Related concepts of interest that weren't mentioned include ideas that generalize hypercyclicity, such as multi-hypercyclicity, supercyclicity, cyclicity and universality. In this paper, I have only skimmed the surface on the applications of hypercyclicity as well as related concepts such as topological transitivity.

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