Absolute Convergence in Ordered Fields

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Abstract

This paper reviews *Absolute Convergence in Ordered Fields* by Clark and Diepeveen [1].

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1 Introduction

In *Absolute Convergence in Ordered Fields* [1], the authors attempt to distinguish between convergence and absolute convergence in ordered fields. In particular, Archimedean and non-Archimedean fields (to be defined later) are examined. In each field, the possibilities for absolute convergence with and without convergence are considered. Ultimately, the paper [1] attempts to offer conditions on various fields that guarantee if a series is convergent or not in that field if it is absolutely convergent. The results end up exposing a reliance upon sequential completeness in a field for any statements on the behavior of convergence in relation to absolute convergence to be made, and vice versa. The paper makes a noted attempt to be readable for a variety of mathematic levels, explaining new topics and ideas that might be confusing along the way. To understand the paper, only a basic understanding of series and convergence in $\mathbb{R}$ is required, although having a basic understanding of ordered fields would be ideal.

2 Definition of Terms

2.1 Field

A field is defined to be a set F with operators $+$ and $\cdot$ that satisfy a variety of conditions. For F to be a field, its operators must be associative and commutative so that

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \text{ and } x + y = y + x \text{ for any } x, y, z \in F.$$

Also, for every $x \in F$ such that $x \neq 0$ there must exist a multiplicative identity and an additive identity, say $x_1, x_2 \in F$ respectively, so that

$$x \cdot x_1 = x \text{ and } x + x_2 = x.$$

In the real numbers, we know $x_1 = 1$ and $x_2 = 0$. There also must exist multiplicative and additive inverses, say $x_3, x_4 \in F$ respectively, so that

$$x \cdot x_3 = 1 \text{ and } x + x_4 = 0.$$

We know for the real numbers that $x_3 = \frac{1}{x}$ and $x_4 = -x$.

2.2 Ordered Field

We will define $F^+ \subset F$ so that,

$$0 \notin F^+,\ x + y \in F^+ \text{ and } x \cdot y \in F^+ \text{ for any } x, y \in F^+.$$

We will define $F^- \subset F$ so that,

$$F^- = \{x : -x \in F^+\}.$$

You may note that F^+ is the set of all positive numbers in F and F^- is the set of all negative numbers in F. Note also that $F = F^+ \cup F^- \cup \{0\}$.

An ordered field is a field that can be ordered. In simplest terms, this requires two conditions to be met. The first condition that must be met is that for any $x \in F$,

$$x \in F^+, x \in F^- \text{ or } x = 0.$$

Secondly, inequalities must be well defined in the ordered field. To avoid unnecessary repetition, $<$ will be defined in a field F with the other inequalities $(\leq, >, \geq)$ defined in a similar manner. For $x, y \in F$, if $x < y$ then,

$$y - x \in F^+ \text{ and } x - y \in F^-.$$

Note $x - y = 0$ if and only if $x = y$.

2.3 The Symbol $\ll$

The symbol $\ll$ will be defined in the following sense. If $x \ll y$ for $x, y \in F$ where F is an ordered field, then $n|x| < |y|$ for all $n \in \mathbb{Z}^+$ where $\mathbb{Z}^+$ denotes the positive integers. Note for this to be true for any $n \in \mathbb{Z}^+$ then x must be an infinitesimal number or y must be an infinite number. In either case, one of the numbers must belong to the hyperreals (numbers constructed using infinites and infinitesimals). A well known example of x would be ϵ.

2.4 Sequential Completeness and Cauchy Sequences

The authors define a sequentially complete field F as a field in which every Cauchy sequence converges. As a reminder, a Cauchy sequence is a sequence where the elements become arbitrarily close to one another as the sequence goes to infinity. Mathematically, a Cauchy sequence $\{a_n\}_{n=1}^{\infty}$ satisfies:

$$|a_n - a_m| < \epsilon \text{ for all } n, m \geq N \text{ for some } N > 0.$$

2.5 New Sequences

The authors define five sequences to be used throughout the paper.

1. A sign sequence is a sequence $\{s_n\}_{n=1}^{\infty}$ with $s_n = \{\pm 1\}$.
2. A Z-sequence in F is a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n > 0$ for all n and $a_n \to 0$ as $n \to \infty$.
3. A ZC-sequence is a Z-sequence such that $\sum_{n=1}^{\infty} a_n$ converges.
4. A ZD-sequence is a Z-sequence such that $\sum_{n=1}^{\infty} a_n$ diverges.
5. A test sequence is a Z-sequence $\{\epsilon_n\}_{n=1}^{\infty}$ such that $\epsilon_{n+1} \ll \epsilon_n$ for all $n \in Z^+$.

3 Summary of Results

In order to prove the main result of the paper (aptly named the Main Theorem), the authors first prove a variety of results. The properties of sign sequences are used to show that if $F \in \mathbb{R}$ is a proper subfield (F is strictly contained in $\mathbb{R}$) then there is a series $\sum_{n=1}^{\infty} a_n$ with terms in F that is absolutely convergent but not convergent. Sign sequences are also used to show that if $\{a_n\}_{n=1}^{\infty}$ is a positive real sequence with $a_n \to 0$ and divergent then for any $L \in \mathbb{R}$, there is a sign sequence $\{s_n\} \in \{\pm 1\}$ such that $\sum_{n=1}^{\infty} s_n a_n = L$. It is then shown that for any ordered field F the following statements are equivalent:

1. F is first countable (every point admits a countable base of neighborhoods);
2. F has countable cofinality;
3. F admits a convergent sequence that is not eventually constant;
4. F admits a ZC-sequence;
5. F admits a Z-sequence;
6. F admits a Cauchy sequence that is not eventually constant.

Similarly, the two following statements are equivalent for any ordered field F:

1. F admits a test sequence;
2. F is non-Archimedean of countable cofinality.

As are the following two statements:

1. F admits a ZD-sequence;
2. F is Archimedean or is not sequentially complete.

All of these results are used in the proof of the Main Theorem below.

4 Proof of Main Theorem

4.1 Main Theorem

Let F be an ordered field.

Case 1

Suppose F is sequentially complete and Archimedean. Then:

1. Every absolutely convergent series in F is convergent;
2. F admits a convergent series that is not absolutely convergent.

Case 2

Suppose F is sequentially complete and non-Archimedean. Then:

1. A series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $a_n \to 0$. In particular,
2. A series is absolutely convergent if and only if it is convergent.

Case 3

Suppose F is not sequentially complete. Then:

1. F admits an absolutely convergent series that is not convergent;
2. F admits a convergent series that is not absolutely convergent.

4.2 Proof

Case 1

1. Assume that $\sum_{n=1}^{\infty} |a_n|$ converges. Then the partial sums form a Cauchy sequence and

$$\sum_{k=n}^{n+m} |a_k| < \epsilon \text{ for } m \geq 0 \text{ and } n \geq N \text{ for some } N.$$

Thus by the triangle inequality,

$$|\sum_{k=n}^{m+n} a_k| \leq \sum_{k=n}^{m+n} |a_k| < \epsilon$$

and so $\sum_{n=1}^{\infty} a_n$ converges by the Cauchy criterion.

2. As a counter example, the convergent series is presented

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

which clearly does not converge absolutely.

Case 2

1. Let F be a sequentially complete non-Archimedean field. First it must be shown that a series $\sum_{n=1}^{\infty} a_n$ in F is convergent if and only if $a_n \to 0$.

The fact that a convergent series has $a_n \to 0$ follows (as usual) from the fact that a convergent sequence is a Cauchy sequence.

Suppose $a_n \to 0$. If F has uncountable cofinality, then a series $\sum_{n=1}^{\infty} a_n$ converges if and only if $a_n = 0$ for all n large enough, hence if and only if $a_n \to 0$.

Now suppose F has countable cofinality. Then there exists a test sequence $\{\epsilon_n\}_{n=1}^{\infty}$. For $k \in Z^+$, choose N_k such that for all $n \geq N_k$, $|a_n| \leq \epsilon_{k+1}$. Then for all $n \geq N_k$ and $\ell \geq 0$, we have

$$|a_n + a_{n+1} + ... + a_{n+\ell}| \leq |a_n| + ... + |a_{n+\ell}| \leq (\ell + 1)\epsilon_{k+1} < \epsilon_k.$$

Thus the sequence is a Cauchy sequence, and hence convergent because F is sequentially complete.

2. The fact that a series in F is convergent if and only if it is absolutely convergent follows immediately, since $a_n \to 0$ if and only if $|a_n| \to 0$.

Case 3

1. Instead of proving the original statement, the authors opt to prove the following. Assume that F is an ordered field. F is sequentially complete is equivalent to every absolutely convergent series in F converges.

   Sequential completeness implying convergence of absolutely convergent series has already been shown in Case 1. It will now be proven that a field not being sequentially complete implies that not every absolutely convergent series in F converges.

   Let $\{a_n\}_{n=1}^{\infty}$ be a divergent Cauchy sequence in F. Then there exists a ZC-sequence $\{c_k\}_{k=1}^{\infty}$. Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, there is a strictly increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that for all $n \geq n_k$, we have

$$|a_n - a_{n_k}| \leq c_k.$$

   It follows that $|a_{n_{k+1}} - a_{n_k}| < c_k$ for all k. Thus $\{a_{n_k}\}_{k=1}^{\infty}$ is divergent and hence so is $\{a_{n_k} - a_{n_1}\}_{k=1}^{\infty}$.

   For $k \in Z^+$, let

$$d_{2k-1} = \tfrac{a_{n_{k+1}} - a_{n_k} + c_k}{2}$$

   and

$$d_{2k} = \tfrac{a_{n_{k+1}} - a_{n_k} - c_k}{2}.$$

   Then,

$$\textstyle\sum_{i=1}^{k}(d_{2i-1} + d_{2i}) = \sum_{i=1}^{k}(a_{n_{i+1}} - a_{n_i}) = a_{n_{k+1}} - a_{n_1}.$$

This is a divergent subsequence of the sequence of partial sums associated to $\{d_k\}_{k=1}^{\infty}$, and hence $\sum_{k=1}^{\infty} d_k$ diverges.

Since $-c_k < a_{n_{k+1}} - a_{n_k} < c_k$, we have

$$|d_{2k-1}| + |d_{2k}| = d_{2k-1} - d_{2k} = c_k.$$

Hence $\sum_{k=1}^{\infty} |d_k|$ is absolutely convergent but not convergent.

2. It is known that F admits a ZD-sequence $\{d_n\}_{n=1}^{\infty}$.

   For $n \in Z^+$, let

$$a_{2n-1} = \frac{d_n}{2},\ a_{2n} = \frac{-d_n}{2}.$$

   Then for all $n \in Z^+$,

$$0 \leq \sum_{k=1}^{n} a_k \leq \frac{d_{n/2}}{2}.$$

   Thus $\sum_{n=1}^{\infty} a_n$ converges (to 0) but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} d_n$ diverges.

5 Further Applications

5.1 Infinite Series in Ordered Fields

When studying a series, often one of the first questions asked is if the series is convergent or not. In an ordered field, this may or may not always be clear. The results from this paper begin to lay out general convergence guidelines for different ordered fields. For example, given a non-Archimedean, sequentially complete field we now know that any convergent series within it is absolutely convergent and vice versa (see Case 2 of the Main Theorem). Perhaps the calculation to show a series in this field is absolutely convergent is easier then the calculation to show it is convergent. For non-Archimedean, sequentially complete fields testing for either convergence will give the same results. On the other hand, if a field is not sequentially complete, we can draw no conclusions about convergence from absolute convergence and vice versa (see Case 3 from the Main Theorem). The ability to know the implications of one type of convergence on another give clearer results and, in problems, could provide a "next step" to be taken. For example, if trying to prove a specific property of a series in a non-Archimedean, sequentially complete field and it is known that the series converges then it is known that the series is absolutely convergent. Absolute convergence could provide more information to the proof than typical convergence would.

5.2 Implications for Convergence Tests

To determine if a series is convergent, it must first be tested. If we know the general behavior of an absolutely convergent series in a field, in relation to convergence for example, we can begin to formulate general tests. To my knowledge, there are not currently many general "tests" that can be applied to a series to see if it is absolutely convergent or convergent in a generalized field. Clark and Diepeveen offer the beginnings of the information required to form these tests. For example, the test for absolute convergence in a sequentially complete field must also imply that the series is convergent but the test for absolute convergence in a non-sequentially complete field must not imply that the series is convergent as it is not guaranteed that every absolutely convergent series converges. The results in this paper, and others in the same vein, could offer the general information to start trying to create these tests.

5.3 Uniform Convergence in Ordered Fields

Consider the close relationship absolute convergence has to the Weierstrass M-test, which tests for uniform convergence. Uniform convergence is a desirable quality for a sequence to have, often leading to much more interesting results and properties. It begs the question, can absolute convergence be used to test for uniform convergence in new ways or in new fields? Even when testing for uniform convergence of a sequence of functions from the definition, absolute convergence often plays a role. As absolute convergence in fields is studied more closely, we may gain insight into uniform convergence in the same fields.

5.4 Integral Convergence in Ordered Fields

Integration in ordered fields is complicated. Requirements for integration (whether Riemann or otherwise) and integral convergence can be convoluted or unknown in many fields and must be studied carefully [2]. The last connection I wish to point out is between sequence convergence and integral convergence. The statement for the Maclaurin-Cauchy test is given below as a refresher:

Let f be a non-negative, monotone decreasing function defined on the interval $[N, \infty)$. Then,

$\sum_{n=N}^{\infty} f(n)$ converges if and only if $\int_N^{\infty} f(x)dx$ is finite (it converges).

The test also provides bounds for the series if convergent but those will be ignored. If there were a function defined by an integral, the behavior of this function in an ordered field could help be exposed by examining how the sequence behaves in this field. Similarly, we may gain insight into how integrals behave by examining how corresponding series behave.

References

[1] Clark, Pete L.; Diepeveen, Niels J. "Absolute Convergence in Ordered Fields". *American Mathematical Monthly* 121.10 (2014): 909-916.

[2] Olmsted, John M. H. "Riemann Integration in Ordered Fields". *The Two-Year College Mathematics Journal* 4.2 (1973): 34-40.