A Brief Introduction to the Theory of Lebesgue Integration

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1 Introduction

Gonzalez-Velasco's paper "The Lebesgue Integral as a Riemann Integral" provides a non-standard, direct construction to the Lebesgue Integral that is meant to be accessible to the reader who is already familiar with the Riemann Integral. Standard treatment of the theory typically relies on a wealth of basic concepts that must be assimilated first, before the reader can begin to appreciate the power of the Lebesgue Integral. Gonzalez-Velasco chooses instead to define the Lebesgue Integral as a (possibly improper) Riemann Integral, and in doing so, provides a background that is immediately familiar to the reader. The theory will be constructed for functions defined over sets of arbitrary measure, which will necessitate a unique set of proofs, including a unique approach to the Lebesgue Dominated Convergence Theorem, which will emphasize the role of uniform convergence. Our end-goal will be to leave the reader comfortable with approaching simple integrals that were otherwise intractable when constrained to Riemann integration.

2 The Lebesgue Measure and Integrable Functions

Firstly, by a rectangle in $\mathbb{R}^n$ we shall mean the product space of n bounded intervals, be they open, closed, or neither. We denote the volume v of a rectangle R to be the product of the lengths of its component intervals.

Definition 1 The outer measure of a set $E \subset \mathbb{R}^n$ is

$$m(E) := inf \sum v(R_i)$$

where the infimum is taken over all finite or countable collections of open rectangles $\{R_i\}$ such that $E \subset R_i$.

Definition 2 A set $E \subset \mathbb{R}^n$ is called measurable if and only if for any set $S \subset \mathbb{R}^n$ then $m(S) = m(S \cap E) + m(S \cap E^c)$. If E is measurable then we denote $m(E)$ as the Lebesgue measure of E.

We note that from these definitions it follows that

(1) $m(\mathbb{R}^n) = \infty$, and the empty set has Lebesgue measure zero.

(2) A set S is measurable if and only if S^c is measurable.

(3) Lebesgue measure is monotonic, i.e. if E and F are measurable with $F \subset E \Rightarrow m(F) \leq m(E)$.

Theorem 1

(1) If E is a finite or countable set then $m(E) = 0$.

Proof: Let us enumerate the elements of E as $\{a_0, a_1, ...\}$. Given some $\varepsilon > 0$, we cover each a_j with an open set O_j of volume $\frac{\varepsilon}{2^j}$. Therefore, the set $\{a_0, a_1, ...\} \subset \bigcup_{j=1}^{\infty} O_j \leq \varepsilon$, and it is now known that every countable union of open sets is a Lebesgue measurable set.

(2) If $\{E_i\}$ is a finite or countable collection of measurable sets then their union $\bigcup E_i$ is measurable and $m(\bigcup E_i) \leq \sum m(E_i)$. If the E_i are disjoint then equality holds.

(3) If $\{E_i\}$ is a finite or countable collection of measurable sets then their intersection $\bigcap$ is measurable. If we have that $m(E_1) < \infty$ and $E_{i+1} \subset E_i \; \forall \, i$ then $m(E_i) \to m(\bigcap E_i)$ as $i \to \infty$.

(4) If E and F are measurable then $E - F := \{x \in E : x \notin F\}$ is measurable. If $m(E) < \infty$ and $F \subset E$ then we have additivity, i.e. $m(E - F) = m(E) - m(F)$.

(5) Every half space in $\mathbb{R}^n$ is measurable.

(6) Every rectangle in $\mathbb{R}^n$ is measurable, and its Lebesgue measure is its volume.

(7) Open and closed sets are measurable.

Definition 3 A function $f : E \subset \mathbb{R}^n \to \mathbb{R}$ is measurable if and only if E is measurable and for any $y \in \mathbb{R}$ the set $\{x \in E : f(x) > y\}$ is measurable.

Theorem 2 The following statements are equivalent:

(1) For each $y \in \mathbb{R}$ the set $\{x \in E : f(x) > y\}$ is measurable.

(2) For each $y \in \mathbb{R}$ the set $\{x \in E : f(x) \geq y\}$ is measurable.

(3) For each $y \in \mathbb{R}$ the set $\{x \in E : f(x) < y\}$ is measurable.

(4) For each $y \in \mathbb{R}$ the set $\{x \in E : f(x) \leq y\}$ is measurable.

Theorem 3 Let $E \subset \mathbb{R}^n$ be a measurable set.

(1) If $f, g : E \to \mathbb{R}$ are measurable then $f + g$ is measurable.

(2) If $\{f_n\}$ is a sequence of measurable functions on E and if $f_n \to f$ on E then f is measurable.

Egorov's Theorem Let $E \subset \mathbb{R}^n$ be a measurable set and let $\{f_n\}$ be a sequence of measurable functions on E such that $f_n \to f$ on E. For any $\delta > 0$ there is a subset G of E with $m(G) < \delta$ and such that $f_n \to f$ uniformly on $E - G$.

3 The Lebesgue Integral.

Given a function $f : E \subset \mathbb{R}^n \to \mathbb{R}$, we first partition its range (as opposed to Riemann integration, where we partition the domain of f). Consider measurable set E where $m(E) < \infty$ and a measurable function $f : E \to \mathbb{R}$ whose values are between zero and $M > 0$. If $0 = y_0 < y_1 < ... < y_k = M$ is a partition of $[0, M]$, and if we define $S_i = \{x \in E : f(x) > y_i\}$ for $i = 0, ..., k$, then f is said to be integrable on E if and only if

$$sup \sum_{i=1}^{k} m(S_i)(y_i - y_{i-1}) = inf \sum_{i=1}^{k} m(S_{i-1})(y_i - y_{i-1}),$$

where the supremum and infimum are taken over all partitions of $[0, M]$. When equality is attained, we will denote the common value by $\int_E f$.

Definition 4. Let $E \subset \mathbb{R}^n$ be a measurable set and let $f : E \to \mathbb{R}$ be a measurable function. The measure function of f on E is the function μ_f defined by

$$\mu_f = \begin{cases} m\{x \in E : f(x) > y\} & \text{if } y > 0 \\ -\, m\{x \in E : f(x) > y\} & \text{if } y < 0 \end{cases}$$

To provide motivation for the definition of our measure function, consider $E \subset \mathbb{R}^n$ where $m(E) < \infty$. Then consider nonnegative measurable function $f : E \to \mathbb{R}$ with bound M. Given partition $0 = y_0 < y_1 < ... < y_k = M$ of $[0, M]$, and noticing that $m(S_i) = \mu_f(y_i)$ with μ_f decreasing, we have that

$$\sum_{i=1}^{k} m(S_i)(y_i - y_{i-1}) \quad \text{and} \quad \sum_{i=1}^{k} m(S_{i-1})(y_i - y_{i-1}),$$

are the lower and upper Riemann sums of μ_f, respectively. Because μ_f is Riemann integrable on $[0, M]$, we have that

$$\int_E f \;=\; R\int_0^M \mu_f,$$

where R indicates Riemann Integral. Furthermore, if instead f is nonpositive, we simply take its integral on E to equal the opposite of the integral $-f$, so

$$\int_E f = -R\int_0^M \mu_{-f} = R\int_{-M}^0 \mu_{-f} = R\int_{-M}^0 -\mu_{-f}(-y)dy\ ,$$

and we have a capable measure function. If $m(E) = \infty$, then our measure function can have infinite values, but otherwise it is Riemann integrable on every bounded subinterval of $(-\infty, 0)$ or $(0, \infty)$, and then the improper integrals

$$R\int_{-\infty}^0 \mu_f \;= -\infty \qquad \text{or} \qquad R\int_0^\infty \mu_f \;= \infty$$

exist.

Definition 5. Let $E \subset \mathbb{R}^n$ be a measurable set, $f : E \to \mathbb{R}$ a measurable function, and μ_f the measure function of f on E. Then the Lebesgue integral of f on E is defined to be

$$\int_E f \;=\; R\int_{-\infty}^0 \mu_f \;+\; R\int_0^\infty \mu_f \;=\; R\int_{-\infty}^\infty \mu_f$$

if the right-hand side exists. If the right-hand side is finite then f is said to be Lebesgue integrable on E and we write $f \in L(E)$.

We can now proceed to demonstrate the use of the Lebesgue Integral, first with two standard improper Riemann Integrals, and third with an integral that is otherwise intractable when constrained to Riemann integration.

Example 1 If $E = \{x \in \mathbb{R} : 0 < x < 1\}$ and $f(x) = \dfrac{1}{x}$, then $\mu_f(y) = 0$ if $y < 0$, $\mu_f(y) = 1$ if $0 < y < 1$, and $\mu_f(y) = \dfrac{1}{y}$ if $y \geq 1$. So we have

$$\int_E f \;=\; R\int_0^1 dy \;+\; R\int_1^\infty \frac{1}{y}dy = \infty.$$

Example 2 If $E = \{x \in \mathbb{R} : 0 < x < 1\}$ and $f(x) = \dfrac{1}{\sqrt{x}}$, then $\mu_f(y) = 0$ if $y < 0$, $\mu_f(y) = 1$ if $0 < y < 1$, and $\mu_f(y) = \dfrac{1}{y^2}$ if $y \geq 1$. So we have

$$\int_E f \;=\; R\int_0^1 dy \;+\; R\int_1^\infty \frac{1}{y^2}dy = 2.$$

Example 3 If $E = \{x \in \mathbb{R} : 0 < x < 1\}$ and

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

then $\mu_f(y) = 0$ for all $y \neq 0$ and we have $R\int_E f \;=\; 0$.

4 Basic Properties of the Lebesgue Integral

Here we will establish some basic properties of our measure function μ_f. In the case that we are operating over some specific domain E of f, then we shall write μ_f^E instead.

Theorem 5 Let $E, F \subset \mathbb{R}^n$ be measurable sets, let $f, g : E \to \mathbb{R}^n$ be measurable functions and let $c \in \mathbb{R}$. Then if the integrals below exist,

(1) $f \leq g \Rightarrow \int_E f \;\leq\; \int_E g$. In the case that we also have $f \geq 0$,

then $g \in L(E) \Rightarrow f \in L(E)$.

(2) $\int_E cf \;=\; c\int_E f$ and then $f \in L(E) \Rightarrow cf \in L(E)$.

(3) If $F \subset E$ then $f \in L(E) \Rightarrow f \in L(F)$. If in addition $f \geq 0$ then $\int_F f \;\leq\; \int_E f$.

(4) If $c > 0$ and if $f^c := min\{f, c\}$ then $\int_E f^c \to \int_E f$ as $c \to \infty$.

Theorem 6 Let $E_1, ..., E_N$ be a finite collection of disjoint measurable sets in $\mathbb{R}^n$, let $E = \bigcup E_i$ and let $f : E \to \mathbb{R}$ be a measurable function. Then

$$\int_E f = \sum \int_{E_i} f \; .$$

Furthermore, if f has a constant value c_i on each E_i, then

$$\int_E f = \sum c_i m(E_i) \; .$$

Theorem 7 Let $E \subset \mathbb{R}^n$ be a measurable set and let $f : E \to \mathbb{R}$ be a measurable function. If $\int_E f$ exists, then

$$\left|\int_E f\right| \leq \int_E |f| \,.$$

Also, $f \in L(E) \Rightarrow |f| \in L(E)$.

5 The Convergence Theorems

Consider a sequence of functions $\{f_n\}$ such that $f_n \to f$. We will demonstrate when it is the case that $\displaystyle\int_E f_n \to \int_E f$. In the case of Riemann integration the uniform convergence of this sequence is sufficient. A similar result can be found for the Lebesgue integral.

Proposition 1. Let $S \subset \mathbb{R}^n$ be a measurable set with $m(S) < \infty$ and let $\{f_n\}$ be a sequence of measurable functions such that $f_n \in L(S)$ and $f_n \to f \in L(S)$ uniformly on S. Then

$$\int_S f_n \;\to\; \int_S f.$$

Proof. Given $\varepsilon > 0$ there is a $K \in \mathbb{Z}^+$ such that $f - \varepsilon \leq f_N \leq f + \varepsilon$, and then $\mu_{f-\varepsilon} \leq \mu_{f_N} \leq \mu_{f+\varepsilon}$, for $N > K$. For $y > 0$ and $N > K$,

$$\begin{aligned}
\mu_f(y+\varepsilon) &= m\{x \in S : f(x) > y + \varepsilon\} \\
&= m\{x \in S : f(x) - \varepsilon > y\} \\
&= \mu_{f-\varepsilon}(y) \\
&\leq \mu_{f_N}(y)
\end{aligned}$$

and then

$$R\!\int_0^\infty \mu_f \;-\; \varepsilon\, m(S) \;\leq\; R\!\int_\varepsilon^\infty \mu_f = R\!\int_0^\infty \mu_f(y+\varepsilon)dy \;\leq\; R\!\int_0^\infty \mu_{f_N}.$$

For $y > \varepsilon$ and $N > K$,

$$\begin{aligned}
\mu_{f_N}(y) \leq \mu_{f+\varepsilon}(y) &= m\{x \in S : f(x) + \varepsilon > y\} \\
&= m\{x \in S : f(x) > y - \varepsilon\} \\
&= \mu_f(y-\varepsilon)
\end{aligned}$$

and then

$$
\begin{aligned}
R\int_0^\infty \mu_{f_N} &\le \varepsilon m(S) + R\int_\varepsilon^\infty \mu_{f_N} \\
&\le \varepsilon m(S) + R\int_\varepsilon^\infty \mu_f(y-\varepsilon)dy \\
&= \varepsilon m(S) + R\int_0^\infty \mu_f \ .
\end{aligned}
$$

Thus, given $\varepsilon > 0$ there is a $K \in \mathbb{Z}^+$ such that

$$
-\varepsilon m(S) \ \le \ R\int_0^\infty \mu_{f_N} - R\int_0^\infty \mu_f \ \le \ \varepsilon m(S)
$$

for $N > K$ and, since ε is arbitrary,

$$
R\int_0^\infty \mu_{f_N} \ \to \ R\int_0^\infty \mu_f
$$

as $N \to \infty$. Similarly,

$$
R\int_{-\infty}^0 \mu_{f_N} \ \to \ R\int_{-\infty}^0 \mu_f
$$

as $N \to \infty$. From these last two limits, and Definition 5, we have established our desired convergence. *Q.E.D.*

The reader should now note the superiority of this facet of Lebesgue theory; rather than requiring the uniform convergence of f_N, we have established convergence with merely the condition that the f_N be uniformly bounded. With said weaker condition we may allow the domains of these functions to have arbitrary Lebesgue measure.

The Lebesgue Dominated Convergence Theorem Let $E \subset \mathbb{R}^n$ be a measurable set and let $\{f_N\}$ be a sequence of integrable functions on E that converges to an integrable function f on E. If there is a function $g \in L(E)$ such that $|f_N| \le g$ for all $\mathbb{N} \in \mathbb{Z}^+$, then

$$
\int_E f_N \ \to \ \int_E f
$$

as $N \to \infty$.

We begin by establishing two lemmas.

Lemma 1 Let $E \subset \mathbb{R}^n$ be a measurable set and let $g \in L(E)$. For any $\varepsilon > 0$

there is a subset F of E with $m(F) < \infty$ such that

$$\Big|\int_{E-F} g\Big| \;<\; \varepsilon \;\; .$$

Lemma 2 Let $F \subset \mathbb{R}^n$ be a measurable set, let $g \in L(F)$ and let G be a subset of F. For any $\varepsilon > 0$ there is a $\delta > 0$ such that if $m(G) < \delta$ then

$$\Big|\int_{G} g\Big| \;<\; \varepsilon \;\; .$$

Because of Lemma 1, we now have that for all functions defined out a set F of finite measure, their integrals are negligible, i.e. less than epsilon. Lemma 2 shows the same is true for a sufficiently small subset G of F. By Egorov's theorem we may choose a set G so that $f_N \to f$ uniformly on $F - G$. With the last Proposition in hand, we will proceed with the proof of the Lebesgue Dominated Convergence Theorem.

Proof: Given $\varepsilon > 0$ define F as in Lemma 1 with respect to E, and choose δ in the spirit of Lemma 2 with respect to ε, and finally choose G as in Egorov's Theorem. By the Triangle Inequality and Theorems 6 and 7,

$$\begin{aligned}
\Big|\int_E f_N - \int_E f\Big| &\le \Big|\int_F f_N - \int_F f\Big| + \int_{E-F} |f_N| + \int_{E-F} |f| \\
&\le \Big|\int_F f_N - \int_F f\Big| + 2\int_{E-F} g \\
&\le \Big|\int_{F-G} f_N - \int_{F-G} f\Big| + \int_G |f_N| + \int_G |f| + 2\int_{E-F} g \\
&\le \Big|\int_{F-G} f_N - \int_{F-G} f\Big| + 2\int_G g + 2\int_{E-F} g \\
&\le \Big|\int_{F-G} f_N - \int_{F-G} f\Big| + 4\varepsilon \; .
\end{aligned}$$

By Proposition 1, we can choose N sufficiently large such that the first term on the right in the last inequality is smaller than ε, and the result follows. *Q.E.D.*

Conclusion The original article was written in a rather terse manner, as is referenced in the very abstract. This summary was meant to include the primary facets of the Lebesgue Integral, including a bare-bones construction of Measure Theory. Though many of the proofs were omitted, it is this author's hope that the reader will have gained insight into a novel approach to Lebesgue integration.

References

Gonzalez-Velasco, Enrique. "The Lebesgue Integral as a Riemann Integral." Internat. J. Math & Math. Sci. Vol. 10 No. 4 (1987) 693-706.