Infinite Products

It’s not obvious what the definition of convergence of infinite products should be. For certain reasons, we don’t want a product $\prod p_n$ to converge to 0. This would be analogous to allowing a sum to converge to $\pm\infty$. So we require that the limit be non-zero. It is convenient to require that the terms $p_n \neq 0$. We can form a partial product $P_m = \prod_{1}^{m} p_n$ and introduce $a_n$ such that $p_n = 1 + a_n$. Then we have the

**Definition 1.** $\prod_{1}^{\infty} p_n$ converges if $\lim_{n \to \infty} P_n = P \neq 0$.

Notice this implies that $p_n \neq 0$. Also this implies that $p_{n+1} = \frac{P_n}{P_{n-1}} \to 1$ and hence that $a_n \to 0$. The proof of the following theorem takes some care. It is not always done correctly. I’m taking this proof from Ahlfors [1].

**Theorem 1.** $\prod_{1}^{\infty} p_n$ converges if and only if $\sum_{1}^{\infty} \log(1 + a_n)$ converges, where $\log(1 + a_n)$ is the principal value of the logarithm. It is not necessarily true that $\log(\prod_{1}^{\infty} P_n) = S$.

**Proof.** If $\sum_{1}^{\infty} \log(1 + a_n)$ converges and is equal to $S$, then it follows, by exponentiating, that $e^S = e^{(\sum_{n=1}^{\infty} \log(1+a_n))} = \lim_{n \to \infty} P_n$. This is the easy part.

Now assume that $\prod_{1}^{\infty} (1 + a_n)$ converges to $P$. Let $S_n = \sum_{m=1}^{n} \log(1 + a_m)$. Then $\log(P_{n+1}) \to 0$ and $\log(1 + a_n) \to 0$. There is the following relation between the principal values of the logarithms,

\[
\log(P_{n+1}) = S_n - (P_n) + 2\pi i q_{n+1} \tag{1}
\]

\[
\log(P_n) = S_n - (P_n) + 2\pi i q_n, \tag{2}
\]

where $q_n, q_{n+1} \in \mathbb{Z}$. Subtract equation (2) from (1) to get

\[
\log(P_{n+1}) - \log(P_n) = \log(1 + a_{n+1}) + 2\pi i (q_{n+1} - q_n).
\]

This equation implies that $(q_{n+1} - q_n) \to 0$ and since the the $q_n$ are integers $q_n = q_{n+1} = q$ for all $n > N$. From equation (2), we conclude that $S_n$ converges to $\log(P) - 2\pi i q$. In other words

\[
\sum_{1}^{\infty} \log(1 + a_n) = \log(\prod_{1}^{\infty} (1 + a_n)) - 2\pi i q.
\]

\[\square\]

**References**