The Mandelbrot Set and the Farey Tree

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Contents

1 Introduction

The Mandelbrot Set, the Farey Tree, and the Fibonacci Sequence [2] explores the relationship between the size of the "limbs" of the Mandelbrot set and the Farey tree. The paper begins by explaining several basics ideas about the Mandelbrot set. Then, Devaney describes two "folk theorems". These theorems are

1. The $\frac{p}{q}$ bulb can be recognized by locating the smallest spoke in the antenna and determining its relative location to the main spoke.

2. The size of bulbs in the Mandelbrot set is determined by the Farey Tree.

The second theorem is the primary focus of the paper and is proved gradually while a geometric explanation of the first theorem is given and not thoroughly proved. In order to complete his proof of the second theorem, Devaney first establishes a mathematical background involving the concept of the Farey Tree, mediant, mathematical specifics about the Mandelbrot Set, and an angle doubling function in relation to the aforementioned topics. These ideas are necessary for the following explanation of several important results of Douady and Hubbard in relation to the external rays which land on the Mandelbrot set. Finally, Devaney turns to proving a version of the second folk theorem and ends with a note about the appearance of the Fibonacci sequence in the Mandelbrot set.

2 The Mandelbrot Set

2.1 Definitions of the Mandelbrot Set

The Mandelbrot Set (Figure 1), is composed of bounded iterates of the critical point of a quadratic polynomial, namely

$$P_c(z) = z^2 + c, \qquad z \in \mathbb{C}.$$

The above polynomical defines a Julia Set (for more information see [6]) and the only critical point occurs at $z = 0$. Now if $P_c(0) = c$ and $P_c^2(0) = c^2 + c$ and so on, the Mandelbrot consists of $c \in \mathbb{C}$ such that $P_c^k(0)$ are bounded [6]. We call this sequence of values the *orbit* of 0. Note that an orbit that returns to itself after n iterations is called a *cycle*. We call a cycle an *attracting cycle* [6] when

$$|P_c^n(0)| < 1.$$

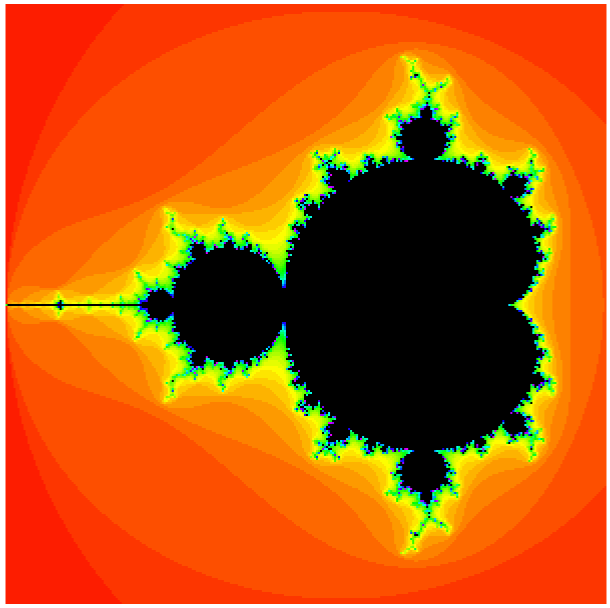

Figure 1: The Mandelbrot Set [4]

Alternatively, we call a cycle a *repelling cycle* [6] when

$$|P_c^n(0)| > 1.$$

Also note that a fixed point z_0 of f occurs when $f(z_0) = z_0$.
The Mandelbrot Set is geometrically composed of a large cardioid shaped section with decorations attached to the *principle cardioid*. We call the decorations that consist of a filled circle attached to the principle cardioid with additional decorations attached *bulbs*. Attached to each bulb is an antenna-like decoration, called the *main spoke*, from which emanates other spokes. The principle cardioid consists of the $c's$ for which $P_c(z)$ has an *attracting fixed point*. We say a fixed point is attracting when $|P_c'(z)| < 1$ [6]. Each of the bulbs attached to the main cardioid contains values of c for which P_c^n has an attracting cycle.

Definition 1. *The rotation number of a bulb is $\frac{p}{q}$ where p is the period of the attracting cycle of the bulb and q is geometrically the number of spokes emanating from the main spoke [3].*

Definition 2. *The $\frac{p}{q}$ bulb is the bulb with rotation number $\frac{p}{q}$.*

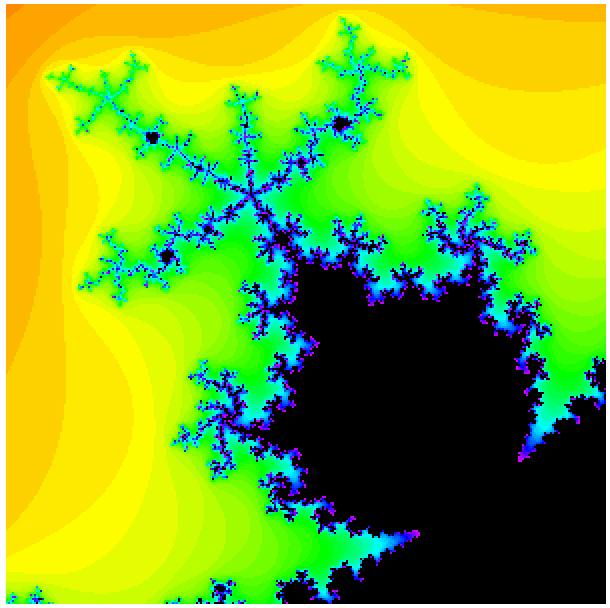

Figure 2: The $\frac{2}{5}$ bulb of the Mandelbrot Set [4]

2.2 The First Folk Theorem

The first theorem presented by Devaney is that

Theorem 2.1. *The $\frac{p}{q}$ bulb can be recognized by locating the smallest spoke and approximating its angle from the main spoke [2].*

Geometrically we see that this should make sense. To obtain q we count the number of spokes, including the main spoke emanating from the bulb. We then obtain p by counting the number of spokes from the main spoke (exclusive) that it takes to reach the smallest spoke, moving counter-clockwise. For example, in the $\frac{2}{5}$ bulb there are 5 spokes and the smallest is located 2 spokes counter-clockwise from the main spoke. That is, the smallest spoke is approximately $\frac{2}{5}$ a turn from the main spoke (Figure 2). Note however that in order to make the notion of "smallness" precise and to prove this theorem true for all bulbs we would need to use hyperbolic measure, which will not be explored here [4].

3 The Farey Diagram and the Mediant

3.1 Farey Diagram

In order to understand the Farey Tree, the concept of mediant must first be explored. The mediant of two rational numbers $\frac{p_0}{q_0}$ and $\frac{p_1}{q_1}$ is defined as

$$\frac{p_0}{q_0} \oplus \frac{p_1}{q_1} = \frac{p_0 + p_1}{q_0 + q_1}.$$

Note that if the two fractions have no number with a smaller denominator

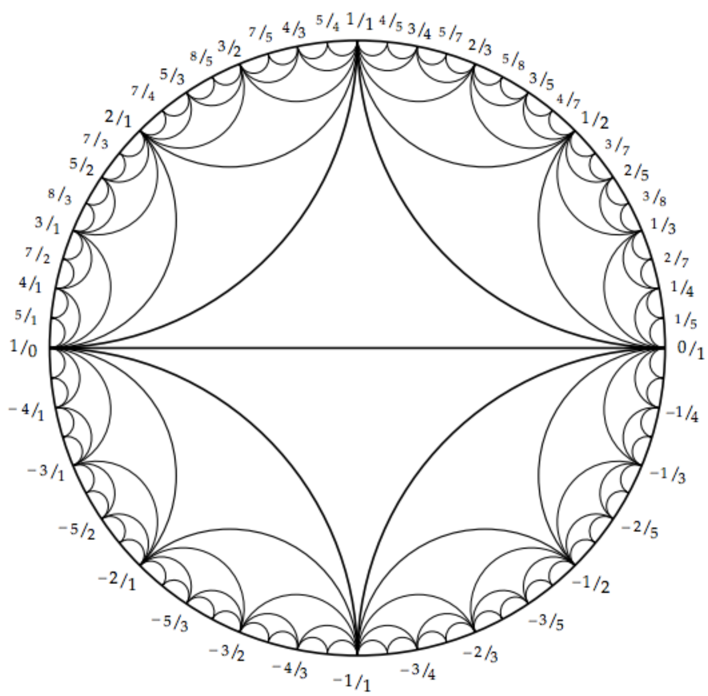

Figure 3: The Farey Diagram [7]

than either of their denominators between them then the resulting fraction is the fraction with the smallest denominator between them. For example, consider the mediant of $\frac{1}{2}$ and $\frac{1}{3}$. There is no multiple of $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{4}$ between $\frac{1}{2}$ and $\frac{1}{3}$. So then

$$\frac{1}{2} \oplus \frac{1}{3} = \frac{2}{5}$$

which has the next smallest denominator, 5.

The mediant is used to construct the Farey Diagram (Figure 3) as follows. First draw a circle and place the following numbers at the coordinates listed below (in terms of the angle in radians):

$$\frac{0}{1} \qquad at\ 0 \tag{1}$$
$$\frac{1}{1} \qquad at\ \frac{\pi}{2} \tag{2}$$
$$\frac{1}{0} \qquad at\ \pi \tag{3}$$
$$\frac{-1}{1} \qquad at\ 2\pi. \tag{4}$$

Note that $\frac{1}{0}$ represents infinity and cannot actually be evaluated. Now, draw a line connecting $\frac{0}{1}$ and $\frac{1}{0}$, and draw curved arcs connecting (1) to (2), (2) to (3), (3) to (4), and (4) to (1). The rest of the diagram is constructed by taking the mediant m of two numbers a and b on the diagram, placing m at the appropriate place on the diagram relative to the other numbers already there, and drawing curved arcs connecting a to m and b to m [7]. For example,

$$\frac{0}{1} \oplus \frac{1}{1} = \frac{1}{2}.$$

So we can then draw curved arcs connecting $\frac{0}{1}$ to $\frac{1}{2}$ and $\frac{1}{1}$ to $\frac{1}{2}$.

Devaney also introduces the terminology of *Farey neighbors* and *Farey child*. Farey neighbors are adjacent rational numbers and a Farey child is the mediant of two adjacent rational numbers. The adjacent rational numbers which produce a Farey child are called its *Farey parents* [2].

Now it is clear the mediant property can be seen geometrically in the Mandelbrot set. Taking the mediant of the rotation numbers, as defined in section 2, of two bulbs of the Mandelbrot set will give the rotation number of a bulb located between them (Figure 4).

The following lemma is also important and will be used in future proofs.

Lemma 3.1. *[2] If* $\frac{\alpha}{\beta}$ *and* $\frac{\gamma}{\delta}$ *are Farey neighbors, and so* $\alpha\delta - \gamma\beta = \pm 1$*, then*

$$\left|\frac{\alpha}{\beta} - \frac{\gamma}{\delta}\right| = \frac{1}{\beta\delta}. \tag{5}$$

We will not give a proof of this lemma here.

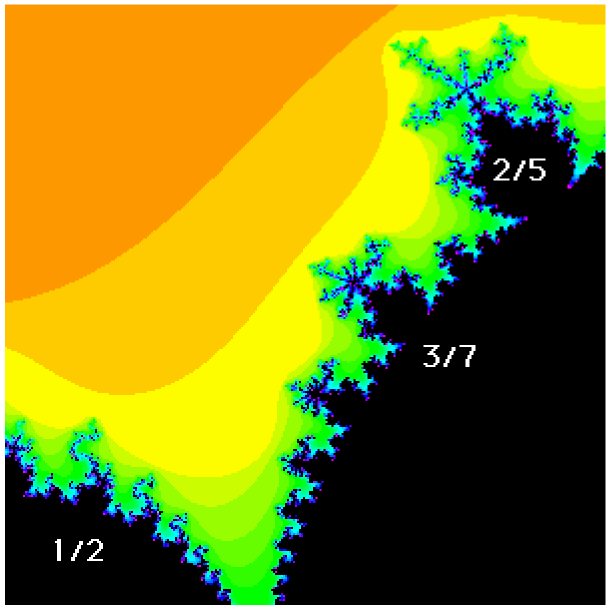

Figure 4: The mediant of the $\frac{2}{5}$ bulb and the $\frac{1}{2}$ bulb [4]

3.2 The Fibonacci Sequence and the Farey Diagram

It is interesting to note that the Fibonacci sequence can be found along a certain portion of the Farey Diagram. The sequence can be constructed by beginning with the rational numbers $\frac{0}{1}$ and $\frac{1}{2}$ and taking the mediant to produce the following sequence:

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{3}.$$

Then taking the mediant of the previous two terms, as one would in the normal construction of the Fibonacci sequence gives

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}.$$

Repeating the process several times begins to generate the Fibonacci sequence in both the numerator and the denominator

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13} \cdots$$

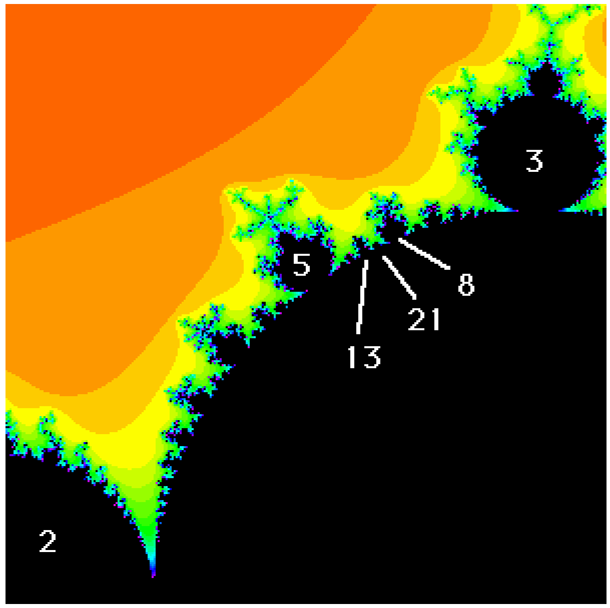

Figure 5: The Fibonacci Sequence in the Size of the Bulbs of the Mandelbrot Set [2]. The numbers denote the period of the particular bulb.

where the denominator is lacking an initial 0, 1. Since the Farey diagram relates to the rotation numbers of bulbs and to the Fibonacci sequence it should not be surprising that the Fibonacci sequence occurs in the Mandelbrot set (Figure 5). We will prove this fact later.

4 Binary Numbers and Itineraries

4.1 Binary Basics

We now move on to a discussion of the ways of finding binary representations of numbers, specifically fractions. The reader is likely familiar with the methods of representing integers as binary numbers. For example, the number 5_{10} in conventional base-ten notation is represented in binary as 110_2. More important to this paper, however, is how to represent fractions as binary numbers.

As an example, consider the fraction $\frac{5}{8}$. In base-ten its decimal expansion is [1]

$$0.625_{10} = 6 \cdot 10^{-1} + 2 \cdot 10^{-2} + 5 \cdot 10^{-3}$$

However, in binary $\frac{5}{8}$ is represented by $.101_2$ which is given by [1]

$$.101_2 = 1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3}$$

The important thing to notice here is that the notation differs only in the base of the power that is used.

4.2 Binary Numbers from the Farey tree

Now we will discuss a method of finding the binary representations of fractions. In order to do so we must first discuss the *doubling function.* The *doubling function* is defined on the real numbers modulo one and is given by $D(\theta) = 2\theta \bmod 1$. Note that θ is periodic (it will repeat itself) under D if and only if θ is of the form $\frac{p}{q}$ which is in lowest terms with q odd [2]. As an example consider $\frac{1}{7}$ which has the following orbit which we can clearly see repeats

$$\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7} \to \frac{1}{7} \to ...$$

However, notice that for $\frac{1}{6}$, the orbit does not ever repeat

$$\frac{1}{6} \to \frac{1}{3} \to \frac{2}{3} \to \frac{1}{3} \to \frac{2}{3} \to \frac{1}{3} \to ...$$

Definition 3. *The itinerary of θ, $B(\theta)$, is its binary expansion.*

The itinerary can be found in the following way as described by Devaney [2]. First, partition a circle where 0 and 1 both lie at the point on the circle with angle 0 radians and $\frac{1}{2}$ lies at the point on the circle with angle π radians. Let θ be a point on the circle in relation to the previous points. Let I_0 denote the upper half of the circle for $0 \leq \theta < \frac{1}{2}$ and let I_1 denote the lower half of the circle for $\frac{1}{2} \leq \theta < 1$. Now choose θ such that it is in lowest terms and has an odd denominator. Thus its orbit is periodic under the doubling function. Now consider the first cycle of the orbit. That is, consider the portion of the orbit that occurs before repetition. For each number s_i in the cycle, if $s_i \in I_0$ assign it a 0 and if $s_i \in I_1$ assign it a 1. Then this sequence of zeros and ones makes up the infinitely repeated segment of the itinerary of θ [2].
As an example, consider $\frac{1}{7}$. It is certainly in lowest terms with an odd

denominator so we can find its itinerary in the previously described manner. From the first part of the cycle before repetition we see that

$$\frac{1}{7}, \frac{2}{7} \in I_0, \qquad \frac{4}{7} \in I_1$$

So $B(\theta) = .\overline{001}$.

5 External Rays and the Mandelbrot Set

Another idea we must consider before proving the second folk theorem stated by Devaney is the concept of external rays and how they land on the Mandelbrot Set. To do so we will explore several results of Douady and Hubbard. The most important of these is that there is a unique analytic isomorphism ϕ which maps the exterior of the unit disk $E = \{z : |z| > 1\}$ to the exterior of the Mandelbrot set [2]. We can see this from the Riemann mapping theorem and a theorem of Carathéodory (see [5] for more details). This map is important because straight rays under ϕ are external rays on the Mandelbrot set with an external angle θ_0 such that $\theta = \theta_0$ where θ is the ray.

Definition 4. *A ray lands when* $lim_{r \to 1} \phi(re^{2\pi i \theta_0 |})$ *exists [5].*

It has been proved that every ray on the Mandelbrot set with a rational external angle lands. This now leads to a result which will be required in the proof of the second folk theorem.

Theorem 5.1. *Suppose a bulb B consists of c-values for which the quadratic map has an attracting q-cycle. Then the root point of this bulb is the landing point of exactly 2 rays, and the angles of each of these rays have period q under doubling [2].*

We will use this theorem without proof. However, the important result is that for every $\frac{p}{q}$ bulb on the Mandelbrot set there are exactly two rays with period q under doubling which land at the root point of the bulb (where it connects to the principle cardioid). For example, there are two rays with period 3 under doubling, $\frac{1}{7}$ and $\frac{2}{7}$, which then land at the root point of the $\frac{1}{3}$ bulb (the largest bulb directly on top of the Mandelbrot set).

6 The Second Folk Theorem

Now that we have established sufficient background material we can make precise the second folk theorem stated in the introduction. We intend to measure the size of limbs of the Mandelbrot set. We define the $\frac{p}{q}$ *limb* to be the portion of the Mandelbrot containing the $\frac{p}{q}$ bulb split from the main cardioid at the root point of the $\frac{p}{q}$ bulb [2]. This will allow us to measure the size of the bulbs of the Mandelbrot set with one additional piece of information, we need to know how to measure the angles of the rays at their landing points.

6.1 Measuring angles

We will use itineraries similar to those discussed in section 4. First, we define

$$R_{(p/q)}(\theta) = e^{2\pi i(\theta + p/q)}$$

where $R_{(p/q)}$ denotes the rotation of the unit circle through $\frac{p}{q}$ turns. Since the angles of the rays will be symmetric about the bulb they will be denoted by $\overline{l_-(p/q)}$, the angle of the ray to the left of the bulb, and $\overline{l_+(p/q)}$, the angle of the ray to the right of the bulb. Here $l_\pm(p/q)$ is a finite string of q digits either 0 or 1 and $\overline{l_\pm(p/q)}$ is the infinitely repeating sequence of $l_\pm(p/q)$ [2]. To find $l_-(p/q)$ we partition the circle similarly to section 4:

$$I_0^- = (0, 1 - \frac{p}{q}], \qquad I_1^- = (1 - \frac{p}{q}, 1].$$

We can then take the itinerary of $\frac{p}{q}$ using our partition. We will denote this itinerary as $s_-(p/q)$. As an example, consider $s_-(2/5)$. We know that the orbit is

$$\frac{2}{5} \to \frac{4}{5} \to \frac{1}{5} \to \frac{3}{5} \to 1 \to \frac{2}{5} \to ...$$

Note, we use the value 1 in our orbit, rather than 0 which is the result of 1 mod 1, because 1 is actually in our partition, while 0 is not. Now using our parition we can construct the itinerary $s_-(2/5) = 01001$.
To find $l_+(p/q)$ we partition the circle as follows

$$I_0^+ = [0, 1 - \frac{p}{q}), \qquad I_1^+ = [1 - \frac{p}{q}, 1)$$

and then take the itinerary $s_+(p/q)$. Here we will use the value 0 rather than 1 if it appears in our orbit because 0 is now in our partition, while 1 is not. So then $s_+(2/5) = 01010$.
It is now clear that the rays landing at the root point of the $\frac{p}{q}$ bulb are given by $s_-(p/q)$ and $s_+(p/q)$. This allows us to then establish an intermediary result to proving the relationship to between the size of the limbs of the Mandelbrot set and the Farey tree. That is, we will show that the size of

the $\frac{p}{q}$ limb is given by the number of external rays that approach it. That is,

Theorem 6.1. *The size of the* $\frac{p}{q}$ *limb is* $\dfrac{1}{2^q-1}$ *[2]. That is*

$$\overline{s_+(p/q)} - \overline{s_-(p/q)} = \frac{1}{2^q-1}.$$

Proof. First, notice that as a result of the fact that the itineraries are the same except at the endpoints of the interval, which are given by

$$R_{p/q}^{q-2}(p/q) = -\frac{p}{q} \text{ and } R_{p/q}^{q-1}(p/q) = 0$$

(where the superscript indicates the item of the itinerary under R being considered), the two itineraries differ only in their last two digits. Using the discussion of binary fractions in section 4 we can rewrite

$$\overline{s_+(p/q)} = \frac{1}{2^{q-1}} + \frac{1}{2^{2q-1}} + \frac{1}{2^{3q-1}} + ... = 2\sum_{i=1}^{\infty}\left(\frac{1}{2^q}\right)^i$$

by noticing that the itinerary is just a geometric sum. Similarly,

$$\overline{s_-(p/q)} = \frac{1}{2^q} + \frac{1}{2^{2q}} + \frac{1}{2^{3q}} + ... = \sum_{j=1}^{\infty}\left(\frac{1}{2^q}\right)^j.$$

So then using the formula for the geometric sum we have that

$$\begin{aligned}
\overline{s_+(p/q)} - \overline{s_-(p/q)} &= 2\sum_{i=1}^{\infty}\left(\frac{1}{2^q}\right)^i - \sum_{j=1}^{\infty}\left(\frac{1}{2^q}\right)^j \\
&= \frac{1}{2^{q-1}} \cdot \frac{2^q}{2^q-1} - \frac{1}{2^q} \cdot \frac{2^q}{2^q-1} \\
&= \frac{1}{2^q-1}.
\end{aligned}$$

□

6.2 The Theorem

Now that we can determine the size of a limb from the angles of the rays landing at the root point of the bulb, we can precede to the main result of Devaney's paper. The proof requires several propositions the proofs of which are sketched but not given in full detail. First we relate the itineraries $s_-(p/q)$ and $s_+(p/q)$ to the Farey parents of $\frac{p}{q}$.

Proposition 1 1. *Suppose that* $\frac{\alpha}{\beta}$ *and* $\frac{\gamma}{\delta}$ *are the Farey parents of* $\frac{p}{q}$ *and that* $0 < \frac{\alpha}{\beta} < \frac{\gamma}{\delta} < 1$*. Then* $\overline{s_{-}(p/q)}$ *consists of the first* q *digits of the of* $\overline{s_{+}(\alpha/\beta)}$ *and* $\overline{s_{+}(p/q)}$ *consists of the first* q *digits of* $\overline{s_{-}(\gamma/\delta)}$*.*

Proof. We will consider only the case for $s_{+}(p/q)$. Since the rays are symmetric about the root point, the case for $s_{-}(p/q)$ will precede very similarly. Using Lemma 3.1 we can see that

$$\frac{\gamma}{\delta} - \frac{p}{q} = \frac{1}{q\delta}.$$

Furthermore, since $\frac{\gamma}{\delta}$ is a Farey parent of $\frac{p}{q}$ we know that its orbit cycles faster than the orbit of $\frac{p}{q}$. So then since the difference between them is exactly $\frac{1}{q\delta}$ the difference in the orbits increases by $\frac{1}{q\delta}$. Consider $\frac{p}{q} = \frac{2}{5}$ $\frac{\gamma}{\delta} = \frac{1}{2}$ as an example. Then the corresponding orbits are

$$\frac{2}{5} \to \frac{4}{5} \to \frac{1}{5} \to \frac{3}{5} \to 0 \to \frac{2}{5}... \qquad \text{and } \frac{1}{2} \to 1 \to \frac{1}{2} \to ... \tag{6}$$

The difference in the orbits then increases by $\frac{1}{10}$. Now, we can consider the difference in the rotation of the circle under both rational numbers,

$$R^{j}_{\gamma/\delta}\left(\frac{\gamma}{\delta}\right) - R^{j}_{p/q}\left(\frac{p}{q}\right) = \frac{j+1}{q\delta}.$$

So then for $j < q-1$,

$$R^{j}_{\gamma/\delta}\left(\frac{\gamma}{\delta}\right) - R^{j}_{p/q}\left(\frac{p}{q}\right) < \frac{1}{\delta}.$$

We know that the points on the orbit of $\frac{\gamma}{\delta}$ under $R_{\gamma/\delta}$ are $\frac{1}{\delta}$ apart and so since the points are also less than $\frac{1}{\delta}$ from the terms of $R_{p/q}$ we want to choose our itineraries such that the first $q-1$ digits of the itineraries of $\frac{\gamma}{\delta}$ and $\frac{p}{q}$ agree. We know that as we go around the circle in the counterclockwise direction the orbit of $\frac{\gamma}{\delta}$ will always be slightly ahead of the orbit of $\frac{p}{q}$ by less than $\frac{1}{\delta}$ units. So as long as we choose $s_{+}(p/q)$ and $s_{-}(\gamma/\delta)$ the corresponding digits are forced to be the same. As an example, consider again $\frac{\gamma}{\delta} = \frac{2}{5}$ and $\frac{p}{q} = \frac{1}{2}$. Then the corresponding orbits are given in (6). In order for the first items in the orbit of each to produce the same value in the itinerary, we need

$$\frac{2}{5}, \frac{1}{2} \in I_0.$$

Regardless of which partition we choose for $\frac{\gamma}{\delta}$, the first element will be in I_0. However, since $\frac{1}{2} = 1 - \frac{1}{2}$, we must choose $s_-(1/2)$ so that the first element is in I_0^-. Now consider the fourth elements of both orbits. By choosing $s_-(1/2)$ we know then that the fourth element of the orbit of $\frac{1}{2}$, $1 \in I_1^-$. So then we need $\frac{3}{5} \in I_1$ as well. However, since $\frac{3}{5} = 1 - \frac{2}{5}$ we must choose $s_+(2/5)$. Intuitively these choices make sense since we are choosing the external rays that land at the root point of their corresponding bulb which are closest together.

Now consider the case when $j = q-1$. Then we know as stated previously that $R_{p/q} = 0$. So then $R^{q-1}_{\gamma/\delta} = \frac{1}{\delta}$. So then the qth digits agree, since the distance between the itineraries is exactly the distance between the points in the orbit of $\frac{\gamma}{\delta}$. So therefore, the final digit of $s_+(p/q) = 0$ and the final digit $s_-(\gamma/\delta) = 0$ as well. However, this is only the case if $\frac{\gamma}{\delta} \neq 1$. If $\frac{\gamma}{\delta} = 1$ then the qth digit cannot be 0.
The other case can be proved in much the same manner. From Lemma 3.1 we now get that $\frac{p}{q}$ - $\frac{\alpha}{\beta} = \frac{1}{q\beta}$ and again the difference in the orbits increases by $\frac{1}{q\beta}$. The reasoning precedes in almost the same manner except that we must choose $s_+(\alpha/\beta)$ and $s_-(p/q)$ [2]. □

Since Proposition 1.1 does not include the case when one of the Farey parents is either 0 or 1, we must consider a second proposition, which is stated below without proof. Details can be found in [2].

Proposition 1 2. *Suppose that 0 is a Farey parent of* $\frac{p}{q}$ *. Then the* q *digits in the lower itinerary of* $\frac{p}{q}$ *are* $s_-(p/q) = 0.....1$*. If 1 is a Farey parent of* $\frac{p}{q}$ *then* $s_+(p/q) = 1....10$.

Now we can turn to the proof of the second folk theorem,

Theorem 6.2. *Suppose* $\frac{\alpha}{\beta}$ *and* $\frac{\gamma}{\delta}$ *are the Farey parents of* $\frac{p}{q}$ *and that* $0 \leq \frac{\alpha}{\beta} < \frac{\gamma}{\delta} \leq 1$*. Then the size of the* $\frac{p}{q}$ *limb is larger than the size of any other limb between the* $\frac{\alpha}{\beta}$ *and* $\frac{\gamma}{\delta}$ *limbs.*

Proof. We will first assume that neither Farey parent is 0 or 1. The case when one parent is either can be handled separately. Now from Proposition 1.1 we know that $\overline{s_-(p/q)}$ and $\overline{s_+(\alpha/\beta)}$ agree in their first q digits and that $\overline{s_+(p/q)}$ and $\overline{s_-(\gamma/\delta)}$ also agree in their first q digits. Then by Theorem 6.1 we have

$$\overline{s_-(p/q)} - \overline{s_+(\alpha/\beta)} \leq \frac{1}{2^q} \qquad \text{and} \qquad \overline{s_-(\gamma/\delta)} - \overline{s_+(p/q)} \leq \frac{1}{2^q}.$$

Since each of the above inequalities gives the difference in angle between the external rays we can approximate the length of the arc between the $\frac{p}{q}$ limb and either of its parents. So, this length is then less than or equal to $\frac{1}{2^q} > \frac{1}{2^q-1}$. So therefore, since we know explicitly the size of the $\frac{p}{q}$ limb, $\frac{1}{2^q-1}$, we have that it attracts the largest number of rays between its two parents.
The case where one parent is either 0 or 1 can be handled in a similar manner.[2] □

Therefore, we can see that the second folk theorem offered in the introduction can be made precise and proven.

6.3 A Note About the Fibonacci Sequence

The previous result leads to an interesting result about the Fibonacci sequence in the Mandelbrot set. As discussed in Section 3, the bulbs of the Fibonacci sequence can be found using the mediant of the rotation numbers of certain bulbs of the Mandelbrot Set. So then from Theorem 6.2 we then have that not only does the Fibonacci Sequence appear in the Mandelbrot Set, but it is composed of the bulbs which are the largest bulb between two parent bulbs.

7 Conclusion

The geometry of the Mandelbrot set can be explored still further. The technique for measuring portions of the Mandelbrot set used to prove the second folk theorem of Devaney's paper can also be used to "compute" the length of various spokes emanating from the bulbs. It then becomes possible to identify particular bulbs by the length of their spokes. In particular it can be shown that the majority of rays that land on a particular limb actually approach the spokes. The process to do so is similar to that used in Devaney's paper and involves considering the rays that land on a particular point of the limb. The point under consideration is called the junction point and is the intersection of the main spoke coming out of the bulb and the rest of the spokes [2]. These results lend further insight into the beautiful Mandelbrot Set.

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