Math 336 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover through section IV.1 in Gamelin.

1. Let $f(z) = x^2 - y^2 + i\log(x^2 + y^2)$. Find the points at which f is complex differentiable. Find the points at which $g(z) = x - iy$ is complex analytic.

2. Let $f(z) = u(z) + iv(z), u = Re(f(z)), v = Imf((z))$ be analytic on an open connected set Ω. Suppose there are real numbers a, b, c with $a^2 + b^2 \neq 0$ and $au(z) + bv(z) = c$ for all $z \in \Omega$. Prove that f is constant.

3. Suppose that v is the harmonic conjugate of u and u is the harmonic conjugate of v. Show that u and v must be constant.

4. Let u be harmonic on W. Prove that $f(z) = u_x(z) - iu_y(z)$ is harmonic.

5. Let a be a complex number and suppose $|a| < 1$. Let $f(z) = \dfrac{z - a}{1 - \bar{a}z}$. Prove the following statements.

   (a) $|f(z)| < 1$, if $|z| < 1$.

   (b) $|f(z)| = 1$, if $|z| = 1$.

6. Let $f(z) = e^{-z^{-4}}$ if $z \neq 0$, $f(0) = 0$. Prove that f is analytic at z if $z \neq 0$ and that the Cauchy-Riemann equations are satisfied at 0. Is f analytic at 0?

7. Let $z_j = e^{\frac{2\pi ij}{n}}$ denote the n roots of unity. Let $c_j = |1 - z_j|$ be the $n - 1$ chord lengths from 1 to the points $z_j, j = 1, \ldots .n - 1$. Prove that the product $c_1 \cdot c_2 \cdots c_{n-1} = n$. *Hint*: Consider $z^n - 1$.

8. Suppose f is analytic on a connected open set. Assume $f^2 = \overline{f}$. Prove that f is constant. What are the possible values of the constant?

9. You will need to know the definitions of the following terms and statements of the following theorems.

   (a) Modulus (absolute value) and argument of a complex number

   (b) Complex derivative

   (c) Complex analytic function

(d) Cauchy-Riemann equations
(e) Harmonic functions and harmonic conjugate
(f) Complex exponential function and trigonometric functions
(g) Complex logarithm and powers
(h) Linear fractional transformations
(i) Mean value principle
(j) Maximum principle
(k) Complex line integrals

13. There may be homework problems or example problems from the text on the midterm.