Riemann Zeta Function and Prime Number Distribution

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1 Introduction

The Riemann Zeta Function is a function of complex variable which plays an important role in analytic number theory and prime number theorem. The function was first studied by Leonhard Euler as a function of real variable and then extended by Bernhard Riemann to the entire complex plane.

2 Definition of zeta function and Functional Equation

2.1 Definition and Euler Product

The Riemann Zeta function could be defined by either as Dirichlet Series or as an Euler Product.

Definition 1.

For all $s$ such that $s = \sigma + it$ and $\sigma > 1$, the Riemann zeta-function $\zeta(s)$ is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1)$$

For all $s$ such that $\sigma \geq \sigma_0 > 1$, we have that the series always converges uniformly because

$$|\sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}}| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma_0}} \quad (2)$$

Using Weierstrass’s theorem, since all the terms in (1) is analytic, the converging function $\sigma(s)$ is holomorphic for all $s$ such that $Res = \sigma \geq 1$

Historically, the zeta-function arose from the need for an analytic tool to deal with prime numbers, as Euler has found an equation that could give an alternating definition for zeta-function.

Theorem 1. (Euler Product) [2]

For $s = \sigma + it, \sigma > 1$, the following holds:

$$\zeta(s) = \Pi\left(\frac{1}{1 - \frac{1}{p^s}}\right) \quad (3)$$

where the product on the right is taken over all prime numbers $p$.

Proof. Let $X \geq 2$, we define the function $\zeta_X(s)$ as

$$\zeta_X(s) = \Pi(p \leq X)(\frac{1}{1 - \frac{1}{p^s}})$$

Then with each of the factors on the right, as $Res \geq 1$, we can always rewrite the term as a geometric series: $\frac{1}{1-p^s} = \sum_{k=0}^{\infty} \frac{1}{p^{sk}}$ where each geometric series
converges absolutely. Hence, we could multiply term by term and replace (3) with the new equation:

\[ \zeta(s) = \prod_{k=0}^{\infty} \frac{1}{p_{k+1}^s} = \sum_{k=0}^{\infty} \cdots \sum_{k_i=0}^{\infty} \frac{1}{(p_1^{m_1} \cdots p_j^{m_j})^s} \]  

(4)

when \( 2 = p_1 < p_2 < \ldots < p_j \) as \( p_j \) represents all the prime numbers up to \( X \). Using fundamental theorem of arithmetic, we know that for any positive integer \( n \), \( n \) is uniquely determined by

\[ n = p_1^{m_1} \cdots p_j^{m_j} \]

when \( m_1 \ldots m_j \) are nonnegative integers and \( p_1 \ldots p_j \) are prime numbers. Consequently we can take the right side of (4) as the form

\[ \sum_{n \leq X} \frac{1}{n^s} + \sum_{n > X} \frac{1}{n^s} \]

when the \( ' \) stands for the summation over those natural numbers \( n > X \) whose prime divisors are all \( \leq X \). We can give an upper bound of this sum by

\[ \left| \sum_{n > X} \frac{1}{n^s} \right| \leq \sum_{n > X} \frac{1}{n^\sigma} \leq \frac{1}{X^\sigma} + \int_X^\infty \frac{du}{u^\sigma} \leq \frac{\sigma}{\sigma-1} X^{1-\sigma} \]  

(5)

The right formula gives an upper bound of the sum. Now combining the definition of \( \zeta_X(s) \) with equation (4) and (5), we obtain the relation between \( \zeta_X(s) \) and the zeta-function as

\[ \zeta_X(s) = \prod(1 - \frac{1}{p_p^s})^j = \sum_{n \leq X} \frac{1}{n^s} + O(\frac{\sigma}{\sigma-1} X^{1-\sigma}) \]  

(6)

Here \( O(\frac{\sigma}{\sigma-1} X^{1-\sigma}) \) represents the upper bound. As we take the limit as \( X \to \infty \), we have \( X^{1-\sigma} \to 0 \) since \( \sigma > 1 \), hence the upper bound of the difference vanished and we proved

\[ \sum_{n=1}^{\infty} = \prod(1 - \frac{1}{p_p^s}) \]  

(7)

### 2.2 Analytic Continuation

The original way of defining zeta-function as sum of geometric series only works when \( \text{Re}(s) = \sigma > 1 \) as the series converges. But Riemann’s basic philosophy is that analytic functions should be dealt with globally on the complex plane. The way of extending \( \zeta(s) \) to the whole complex plane involves a new function defined by taking a contour integral over a curve \( \gamma \)

\[ \phi(s) = \frac{1}{2\pi i} \int_{\gamma} \frac{(-z)^{s-1}}{e^z - 1} \]  

(8)
when the curve $\gamma$ is defined by following a branch on the top with real part from $\infty$ to $\epsilon$, detouring along a circle of radius $\epsilon$ around the origin, the returning from $\epsilon$ to $\infty$ along the bottom edge of the branch. The integral function converges and is entire on the whole complex plane.

To evaluate that, we split the curve into 3 pieces, and first assume that $\text{Re}(s) > 1$, get the equation

$$\phi(s) = \frac{1}{2\pi i} \left( \int_\infty^\epsilon \frac{e^{(s-1)(\log x - i\pi)}}{e^x - 1} \, dx + \int_{|z|=\epsilon} \frac{(-z)^{s-1}}{e^{-1}} \, dz + \int_\epsilon^\infty \frac{e^{(s-1)(\log x + i\pi)}}{e^x - 1} \, dx \right)$$

(9)

(as the on curve from $\infty$ to $\epsilon$ and from $\epsilon$ to $\infty$ are horizontal lines, $dy = 0$ so we could replace $dz$ with $dx$.) consider the integrand of the second integral, as $e^z - 1 = 0$ on a simple point as $z = 0$, we can use Taylor expansion for $e^z - 1$ and get $\sum_{n=1}^\infty x^n$. By that, we can bound the integrand of the second integral on the circle of $|z| = \epsilon$ by $C * \epsilon^{Re(s)-2}$. Now as $\epsilon \to 0$ we have that the second integral also approaches 0. Passing the limit to the formula, we now obtain (as $y = 0$, we have that $dz = dx$)

$$\phi(s) = \frac{1}{2\pi i} \left( \int_0^\epsilon \frac{x^{s-1}}{e^x - 1} \, dx * (-1)^{s-1} - \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx * (-1)^{s-1} \right)$$

$$= \frac{1}{2\pi i} \left( e^{i\pi(s-1)} - e^{-i\pi(s-1)} \right) \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx$$

(10)

now rewrite the term in the parentheses, we obtain that

$$\phi(s) = -\frac{\sin(\pi s)}{\pi} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx$$

(11)

then treat the integrand as a geometric series,which is justified as we are now under the assumption that $\text{Re}(s) > 1$, we then get that

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx = \int_0^\infty \left( \sum_{n=1}^\infty e^{-nx} \right) x^{s-1} \, dx$$

(12)
as the geometric series converges uniformly for all the interval \([ε, \infty)\), we can justify interchanging the summation and integral sign, and hence get

\[
\int_0^\infty \left( \sum_{n=1}^\infty e^{-nx}x^{s-1} \right) dx = \sum_{n=1}^\infty \int_0^\infty e^{-nx}x^{s-1} dx
\]

\[
= \frac{1}{n^s} \sum_{n=1}^\infty \int_0^\infty e^{-t}t^{s-1} dt = \sum_{n=1}^\infty \frac{\Gamma(s)}{n^s} = \Gamma(s)ζ(s) \tag{13}
\]

for \(Re(s) > 1\). Now substitute this equation to (11), we get the following

\[
φ(s) = -\frac{\sin(\pi s)}{\pi} Γ(1-s)ζ(s) = -\frac{ζ(s)}{Γ(1-s)} \tag{14}
\]

with the property of the Gamma Function. Now we can express \(ζ(s)\) with the Gamma function and \(φ(s)\) which we just made. We obtain

\[
ζ(s) = -Γ(1-s)φ(s) = \frac{Γ(1-s)}{2πi} \int_{γ} \frac{(-z)^{s-1}}{e^z - 1} dz \tag{15}
\]

here, both \(φ(s)\) and \(Γ(s)\) are entire functions except for \(Γ(s)\) has a simple pole at \(s = 0\) with residue 1. Hence, \(ζ(s)\) is analytic on the whole complex plane except for a simple pole at \(s = 1\).

### 2.3 The Functional Equation

The zeta-function has a very nice functional equation which would help people compute its value and analyze it properties. For the zeta-function \(ζ(s)\), we have the functional equation as

\[
ζ(s) = 2^sπ^{s-1} \sin\left(\frac{πs}{2}\right) Γ(1-s)ζ(1-s) \tag{16}
\]

One way to get this functional equation is to slightly modify the curve of the contour integral we use to define \(φ(s)\); we change the whole path from \(n\) going to origin to \(n\) going to infinity, to a rectangle centered at origin with horizontal length \(2n\) and vertical length \(sπn\). We define such a path as \(γ_n\), and define

\[
φ_n(s) = \frac{1}{2πi} \int_{γ_n} \frac{(-z)^{s-1}}{e^z - 1} dz. \tag{17}
\]

in this case, we could generally bound the term \(|e^z - 1|\) on the edge of the rectangle by the following:

- On the horizontal edge, we have that \(|e^z - 1| ≥ |Im(e^z)| = |e^{imπ}| = 1\)
- On the vertical edge, we have that \(|e^z - 1| ≥ \min\{|e^1 - 1|, |e^{-1} - 1|\} > \frac{1}{2}\)

Now we have that \(|e^z - 1| > \frac{1}{2}\) on edges of the rectangle, we could bound the integrand by \(2n^{s-1}\). By ML-estimate, the integral over the edges are bounded by \(Cn^s\). In this case, as \(s < 0\), we have that this integral over edges tends to
0 as $n \to \infty$. Now, the difference between $\int_{\gamma} \psi$ and $\int_{\gamma}$ is the integral around the closed loop with the boundary of the rectangle $R_n$ and a detour along an elongated indentation that doesn’t cross over the branch cut. The poles in this closed contour are at the points of $z = 2\pi ki$ when $0 < k \leq n$, such that $e^z - 1 = 0$. Every pole is a simple pole with residue of

$$\left. \frac{(-z)^{s-1}}{e^z} \right|_{z=\pm 2\pi ki} = (2\pi)^{s-1} |k|^{s-1} e^{(s-1)\log(i)}$$

Now we apply residue theorem with the close contour of $\gamma_n - \gamma$ and obtain that

$$\phi_n(s) - \phi(s) = \sum_{k=1}^{\infty} (2\pi)^{s-1} k^{s-1} * (e^{(s-1)i\pi/2} + e^{-(s-1)i\pi/2})$$

(18)

Now we combine terms and get

$$e^{(s-1)i\pi/2} + e^{-(s-1)i\pi/2} = 2\cos((s - 1)\frac{\pi}{2}) = 2\sin(\frac{\pi s}{2})$$

(19)

Now as know that $\phi_n(s) \to 0$ as $n \to \infty$. With the substitution of formula (19) we get the following equation:

$$-\phi(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \zeta(1 - s)$$

(20)

Combining this with (15), this eventually becomes the final form of functional equation of

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1 - s) \zeta(1 - s)$$

(21)

2.4 Zeros of Riemann zeta-function

The values $s$ when $\zeta(s)$ attains zero are called zeros of Riemann zeta-function. From the functional equation (16), one can easily deduce that $\zeta(s) = 0$ when $s = -2, -4, -6,...$. Those zeros are called trivial zeros since they have much smaller significance.

The rest of zeros, are all at the critical strip in complex plane, which is the strip of complex numbers with real parts $0 < \Re(s) = \sigma < 1$. Those zeros are called non-trivial zeros and their distributions is an important topic in analyzing number theories and prime number distribution. In 1859, Riemann stated a formula in his paper about the number of zeros in critical strip, which was later proved by von Mangoldt in 1905 [1], namely

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

(22)

here $N(t)$ represents the number of zeros in the region $0 < \sigma < 1, 0 < t \leq T$ as $s = \sigma + it$. The precise location of zeros of $\zeta(s)$ still remain to this day an unsolved mystery in math, and the Riemann hypothesis could be recast in the formula $N(T) = N_0(T)$ for all $T > 0$, where $N_0(T)$ is the number of zeros of
The zeta function \( \zeta(s) \) of the form \( s = \frac{1}{2} + it, 0 < t \leq T \), in other words, all non-trivial zeros are on the line with \( \text{Re}(s) = \frac{1}{2} \), which is called the critical line. The first nontrivial result is proved by G.H. Hardy in 1914, which gives

\[
N_0(T) > CT \quad (C > 0, T \geq T_0)
\]  

(23)

This indicates that there are infinitely many zeros on the critical line. The result was improved in 1942 by A. Selberg to

\[
N_0(T) > CT \log T \quad (C > 0, T \geq T_0)
\]  

(24)

based on equation 8, since the number of total zeros on critical strip is bounded, this is already the best lower bound we can give. In 1974, N. Levinson proved in his paper that

\[
N - 0(T + U) - N_0(T) > V(N - 0(T + U) - N(T))
\]  

(25)

for \( C = \frac{1}{3} \), which states that at least \( \frac{1}{3} \) of all non-trivial zeros are on the critical line. Although this was a quite impressive result, it’s still much weaker than Riemann Hypothesis and this method itself seems not to be able to yield the result \( N_0(T) \sim N(T) \) for \( T \to \infty \).

On the other hand, huge effort is also devoted to compute explicitly the non-trivial zeros using computers and try to find counter examples for Riemann hypothesis. [3] The initial computations were done by Cray supercomputer using Riemann-Siegel formula and computed that the first \( 10^{12} \) non-trivial zeros are all on the critical line. Then an improved algorithm, jointly with Arnold Schonhage, helped to compute the first \( 10^{20} \) non-trivial zeros. By far, people has already worked through the first \( 10^{22} \) non-trivial zeros, and no counter example of Riemann hypothesis has been found. The computation is still continued using spare cycles on machines at AT&T labs. There are some statements that the counter examples may lie far away from the values we could compute with current algorithm.

### 3 Prime Number Distribution

#### 3.1 Related Functions and Gauss’ Conjecture

To begin the discussion about the distribution of prime number, we have to start with the definition of several functions that’s related to prime numbers on the real line. First, the prime counting function

\[
\pi(x) = \sum_{p \leq x} 1
\]  

(26)

shows the number of primes not exceeding \( x \). Then, the von Mangoldt function

\[
\Lambda(n) = \begin{cases} 
\log p, & n = p^k \text{ (prime, } k \geq 1) \\
0, & \text{otherwise}
\end{cases}
\]  

(27)
and the more important Chebyshev function which is defined as

\[ \psi(x) = \sum_{n \leq x} \Lambda(n). \]  

(28)

There is a closed connection between \( \pi(x) \) and \( \phi(s) \), which is given by the formula

\[ \pi(x) = \frac{\phi(x)}{\log x} (1 + O\left(\frac{1}{\log x}\right)) \]  

(29)

this equation shows that the prime number counting function \( \pi(x) \) roughly differ the Chebyshev-\( \phi \) function by a factor of \( \log x \). Gauss made an important conjecture with an estimation about prime counting function as

\[ \pi(x) \sim \int_2^x \frac{dt}{\log t} \]  

(30)

where as the formula on the right of (11) is sometimes also denote as function \( \text{Li}(x) \). The prime number theorem, which states that

\[ \phi(x) \sim x \]  

(31)

is roughly equivalent to the estimate that \( \pi(x) \sim \frac{x}{\log x} \).

3.2 Connection between Riemann zeta-function and the distribution of prime numbers.

One convenient way to connect Riemann zeta-function to prime number distribution is to cast the Euler identity (2) into another form by taking the logarithm on both sides and then differentiating, which leads to the formula

\[ -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\log p}{p^s} = \sum_{n=1}^{\infty} \frac{(\log p)p^{-ns}}{n} \]  

(32)

The procedure is justified as \( \text{Res} = \sigma > 1 \) we can always write \( p^s - 1 \) in the form of geometric series. Then with the right formula, change the order of summation, we obtain that for \( \text{Res} = \sigma > 1 \),

\[ -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s} = \int_1^\infty x^{-s} d\psi(s) \]  

(33)

equation (31) is important in the sense of connecting the zeta-function and the distribution of prime numbers. Evaluate this integral by parts, we get that

\[ \frac{\zeta'(s)}{\zeta(s)} = x^{-s}\psi(x)|_1^1 - \int_1^\infty \psi(s)d(x^{-s}) \]  

(34)

by definition, we have that \( \psi(1) = 0 \) and \( \psi(x) = O(x\log x) \) as \( x \to \infty \), we can simplify this equation down to

\[ -\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty x^{-s-1}\psi(s)dx. \]  

(35)
which points a deep connection between the zeta function and the distribution of prime numbers.

### 3.3 Formula for prime number function with zeros in zeta function

There is one formula which links zeros of $\zeta(s)$ to Chebyshev $\psi$ function:

$$
\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O(xT^{-1}(\log xT)^2) + O(\log x) \tag{36}
$$

when $\rho$ denotes zeros for zeta function. To prove this formula, we need to prove first a lemma. [1]

#### Lemma 1.

Let $\delta = 0$ if $0 < y < 1$, $\delta(1) = \frac{1}{2}$, $\delta(y) = 1$ if $y > 1$, and

$$
I(y,T) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} y^s s^{-1} ds \tag{37}
$$

Then for $y,c,T > 0$ we have

$$
|I(y,T) - \delta(y)| < \begin{cases} 
  y^{\min(1,T^{-1}|\log y|^{-1})} & \text{if } y \neq 1, \\
  cT^{-1} & \text{if } y = 1 
\end{cases} \tag{38}
$$

After the Lemma is proved, we could start prove this theorem by set $c = 1 + (\log x)^{-1}$ and consider the following integral

$$
J(x,T) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'(s)x^s}{\zeta(s)s} ds \tag{39}
$$

For $\text{Re}(s) = \sigma > 1$, we can apply equation (32) and get that the series converges absolutely. Then apply Lemma 1, we get the following equation:

$$
\sum_{n \leq x} \Lambda(n) = J(x,T) + O(\sum_{n=1,n \neq x}^\infty \Lambda(n)(x/n)^{c\min(1,T^{-1}|\log x/n|^{-1})}) + O(T^{-1}\log x) \tag{40}
$$

To estimate this series we consider first the terms with $n \leq \frac{3}{4}x$ or $n \geq \frac{5}{4}x$. Then $|\log x/n| \gg 1$, and since $x^c = cx$ the contribution of these term is

$$
\ll xT^{-1} \sum_{1}^{\infty} \Lambda(n)n^{-c} = \frac{xT^{-1}\zeta'(c)}{\zeta(c)} \ll x^{T-1}(c-1)^{-1} = T^{-1}x\log x
$$
Then the work left is to estimate the term with \( \frac{3}{4} < n < \frac{5}{4} \). We get that the terms with in this range have contributions \( \ll T^{-1} x \log^2 x \), while there are at most five terms, the sum of them is \( \ll \log x \). Therefore we obtain

\[
\sum_{n \leq x} \Lambda(n) = J(x, T) + O(T^{-1} x \log^2 x) + O(\log x) \tag{41}
\]

Then the next step of the proof is to replace the vertical segment from \( c -iT \) to \( c + iT \) in \( J(x, T) \) by the other three sides of the rectangle with vertices \( c -iT, c + iT, -U + iT, -U - iT \), where \( U \) is a large odd integer. If \( T \neq \gamma \) for any zero \( \rho \), then the residue theorem gives

\[
\psi(x) = x - \sum_{|\gamma| < T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \sum_{0 < 2m < U} x^{-2m} - 2m + O(T^{-1} x \log^2 x) + O(\log x) + \frac{1}{2\pi i} \int_{\alpha} -\frac{\zeta'(s)x^s ds}{\zeta(s)s} \tag{42}
\]

when \( \alpha \) is the modified curve. Based on the fact that the number of zeros \( \rho \) for which \( -\gamma - T \) \( \ll 1 \) is \( O(\log T) \), the difference of those zeros could not all be \( o(1/\log T) \). Hence we can choose \( T \) so that \( |\gamma - T| \gg (\log T)^{-1} \) for all zeros \( \rho \). The integral in (41) would all contribute to error terms in (36), and we need bounds for \( \frac{\zeta'(s)}{\zeta(s)} \) to estimate the error term, which could be stated as

\[
\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho, |\gamma - T| < 1} \frac{1}{s - \rho} + O(\log T) \ll \log^2 T \quad (-1 \leq \sigma \leq 2) \tag{43}
\]

then use the functional equation, take the derivative to obtain the bound for \( \rho \leq -1 \) we have that

\[
\frac{\zeta'(s)}{\zeta(s)} \ll \log(2|s|) \quad (\sigma \leq -1) \tag{44}
\]

Now using these two bounds we get have

\[
\int_{c \pm iT}^{c \mp iT} \ll T^{-1} \log^2 T \int_{-\infty}^{c} x^\sigma d\sigma \ll x \log^2 T / (T \log x) \tag{45}
\]

while (43) gives

\[
\int_{c \pm iT}^{-U - iT} \ll U^{-1} \log U \int_{-T}^{T} x^{-U} dt \ll T \log U / U x^U = o(1) \quad (U \to \infty) \tag{46}
\]

insert this result to Riemann-von Mangoldt formula (18), we eventually proved that

\[
\sum_{|\gamma| < T} \frac{x^\rho}{\rho} = \sum_{|\gamma| < T + A} \frac{x^\rho}{\rho} + O(x T^{-1} \log T). \tag{47}
\]
3.4 Riemann Hypothesis and Prime Number distribution

Riemann Hypothesis suggests that all the zeros on the critical strip was on the critical line with $\sigma = \frac{1}{2}$, although it’s not been proved yet, it could yield a lot of elegant results and could be written in the equivalent form of some nice bounds for prime numbers distribution.

For example, there is a theorem states that if $\frac{1}{2} \leq \theta < 1$ is fixed, then

$$\psi(x) = x + O(x^\theta \log^2 x)\quad (48)$$

if and only if

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > \theta \quad (49)$$

To prove this equation, we set $\sigma > 1$ and then obtain

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s} = \int_{1^-}^{\infty} y^{-s}d\phi(y) = s \int_{1}^{\infty} \phi(y)y^{-s-1}dy. \quad (50)$$

if we set $\phi(x) = x + R(x)$, we then obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} + s \int_{1}^{\infty} R(y)y^{-s-1}dy \quad (\sigma > 1) \quad (51)$$

Now if $R(x) \ll x^\theta \log^2 x$, then the integral (50) is just a regular function of $s$ for $\sigma > \theta$, so $\zeta(s)$ would not vanish. Then we deduce (48) from (49) using the theorem (36) we just proved. Then for $\rho = \beta + i\gamma$, $|\gamma| \leq T$ we have $\beta \leq \theta$ and choose $T = x^{1-\theta}$ we obtain the following

$$\phi(x) = x + O(x^\theta \sum_{|\gamma| \leq T} \frac{1}{|\gamma|}) + O(xT^{-1} \log^2 x)$$

$$= x + O(x^\theta \log^2 x) + O(xT^{-1} \log^2 x) = x + O(x^\theta \log^2 x) = x + O(x^\theta \log^2 x). \quad (52)$$

Now as we already proved this theorem, with Riemann Hypothesis, we can easily deduce this formula to

$$\psi(x) = x + O(x^{\frac{1}{2}} \log^2 x) \quad (53)$$

which could be a nice bound for the prime number function.
References

