Intertemporal Consumption and Savings Behavior: Neoclassical, Behavioral, and Neuroeconomic Approaches

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Abstract

This paper summarizes neoclassical, behavioral, and neuroeconomic models of intertemporal consumption and savings behavior. I summarize the construction and implications of Modigliani & Brumberg’s Life-Cycle Hypothesis [4] and Laibson’s quasi-hyperbolic consumption function [8] as background and motivation for Bisin & Benhabib’s neuroeconomic model of dynamic consumption behavior [3]. In particular, I focus on the mathematical construction of each model, their different behavioral assumptions of agent rationality, and the resulting implications for economic theory.

1 The Neoclassical Life-Cycle Hypothesis

The development of modern economic theories analyzing intertemporal consumption began in the early 1950s with the publication of Modigliani & Brumberg’s Life-Cycle Hypothesis (LCH) [4]. By making several arguably innocuous assumptions, Modigliani & Brumberg produce several important and non-trivial predictions about macroeconomic processes such as the relationship in aggregate economics between savings and growth. Furthermore, since its construction, the LCH has served as the standard economic approach to the study of consumption and savings behavior and has served as a foundational basis for subsequent models of intertemporal consumption.

The structure of the rest of this chapter is as follows: subsection 1.1 presents the theoretical foundations and assumptions of the LCH. Subsection 1.2 discusses some of the implications and results. Subsection 1.3 concludes the discussion on neoclassical models of intertemporal consumption and discusses several objections and limitations of the LCH.
1.1 Theoretical Foundations

Before we begin the construction of the life-cycle hypothesis, let us first define a few terms that readers unfamiliar with economic theory may not know.

Definition 1.1. Utility: An economic term referring to the total satisfaction received from consumption.

Definition 1.2. Marginal Propensity to Consume (MPC): The proportion of income dedicated to consumption.

Mathematically, the marginal propensity to consume is given as \( \text{MPC} = \frac{\partial Y}{\partial C} \), where \( Y \) represents income and \( C \) is the consumption function. The MPC can thus be thought of how much extra consumption an additional dollar of income induces.

Having defined the above key terms, we now move to the construction of Modigliani’s model. Consider the following variables:

- \( c_t \): The agent’s consumption during time period \( t \)
- \( y_t \): The agent’s income (other than interest) in time period \( t \)
- \( s_t \): The agent’s savings in time period \( t \)
- \( a_t \): The agent’s assets at the beginning of time period \( t \)
- \( U \): The agent’s utility function
- \( r \): The rate of interest
- \( N \): The earning span
- \( M \): The retirement span
- \( L \): The life span, which for this purpose is \( N + M \)

Since individuals prefer to be happier, all else constant, we assume that our given agent tries to maximize expected lifetime utility, subject to some constraints. This leads us to assumption 1.1:

Assumption 1.1. Individuals only receive utility from aggregate consumption in current and future periods.

Mathematically, assumption 1.1 states that \( U = U(c_t, c_{t+1}, ..., c_L) \). Informally, this means that the only things which affect an agent’s utility, or satisfaction level, is their total consumption over the course of their lives. Furthermore, since consumption is constrained by income, our agent faces the following problem:

\[
\max_{c_t, c_{t+1}, ..., c_L} U(c_t, c_{t+1}, ..., c_L) \quad \text{s.t.} \quad \sum_{\tau=t}^{N} \frac{y_{\tau}}{(1+r)^{\tau+1-t}} = \sum_{\tau=t}^{L} \frac{c_{\tau}}{(1+r)^{\tau+1-t}} \tag{1.1}
\]

The constraint in (1.1) is simply the budget constraint – income must equal expenditures.

Assumption 1.2. \( U \) is homogeneous with respect to consumption at different points in time.

Assumption (1.2) is fundamental to the construction of the LCH, and can be restated as follows: if the agent unexpectedly receives a dollar, then they will allocate that dollar to the remaining consumption periods such that the new consumption levels maintain the same proportions. Mathematically, let \( c_\tau \) denote planned future consumption at time \( \tau \). Then assumption (1.2) states that \( c_\tau = \gamma_\tau v_t \), where \( \gamma_\tau \) is a proportionality factor which is dependent on \( U, r, \) and \( t \), but not on total present value of wealth \( v_t \). Furthermore, \( v_t \) is expressed as the sum of the previous period’s net worth, plus income and the present value of future income:

\[
v_t = a_{t-1} + y_t + \sum_{\tau=t+1}^{N} \frac{(y_\tau)^\tau}{(1+r_\tau)^{\tau-t}} \tag{1.2}
\]
Assumption 1.3. Utility is exponentially discounted.

In its simplest form, an exponentially discounted utility means that a util (measure of unit for utility) delayed $\tau$ periods is worth $\delta^\tau$, with $0 < \delta < 1$, as much as a util enjoyed immediately. More formally, as described in [6], the discounted utility model has the following functional form with discount rate $\rho$:

$$U(c_t, ..., c_T) = \sum_{k=0}^{T-t} \left( \frac{1}{1+\rho} \right)^k u(c_{t+k})$$  \hspace{1cm} (1.3)

It is important to note that while (1.3) has a simple, intuitive functional form and is a standard assumption in the economics literature, it has little empirical support in psychology or neuroscience. This topic is discussed more in subsection 1.3 as well as section 2.

It is from these three main assumptions that Modigliani & Brumberg produce the main results of the LCH. Note that while several more assumptions are outlined in [4], they are not necessary and can be relaxed at the expense of mathematical simplicity. Thus, it is remarkable how these simple assumptions can lead to several wide-ranging and important predictions about both microeconomic decision-making processes and macroeconomic dynamics.

1.2 Implications and Results

The LCH produces numerous implications and results, many of which have served as the basis of government policy recommendations. Because of the numerous results, implications, and discussions of the LCH, I restrict this subsection to only a few results relating to consumer and microeconomic theory. For an exhaustive compilation of results, see [1], [4], [5].

Proposition 1.1. Current consumption is a linear and homogeneous function of current income, expected average income, and assets.

More specifically, the consumption function is given as:

$$c = c(y, y^e, a, t) = \frac{1}{L_t}y + \frac{(N-t)}{L_t}y^e + \frac{1}{L_t}a$$  \hspace{1cm} (1.4)

where the undated variables relate to the current period. Equation (1.4) says that consumption at any time period is distributed evenly across their life-time. In other words, agent’s ”smooth” their consumption profile over each stage of their life.

Proposition 1.2. The optimal planned consumption profile appropriates equal subdivisions of expected aggregate income to consumption in each period.

Mathematically, let $\bar{c}_1^\tau$ denote the consumption plan at time 1. Furthermore, assume that the agent expects a constant income $y_1$ throughout $N$. Then to maximize lifetime utility subject to income constraints, the optimal planned consumption profile is:

$$\bar{c}_1^\tau = \frac{N}{L}y_1, \quad \tau = 1, 2, ..., L$$  \hspace{1cm} (1.5)
**Proposition 1.3.** The optimal savings plan smooths consumption.

Mathematically, the optimal savings plan is:

\[
s^1_\tau = \begin{cases} 
\frac{M}{L} y_1, & \tau = 1, 2, ..., N \\
\frac{-N}{L} y_1, & \tau = N + 1, N + 2, ... L 
\end{cases}
\]

(1.6)

See figure 1 for a graphical depiction of Propositions 1.1-1.3.

Together, propositions 1.1-1.3 form the standard microeconomic model of intertemporal consumption and savings: rational agents with exponentially discounted utility maximize their lifetime utility by choosing feasible consumption and savings plans such that their consumption profiles are "smoothed" over the course of their lifetime.

Figure 1: The Life-Cycle Savings Hypothesis
1.3 Discussion and Limitations

While the neoclassical LCH model has served as the standard economic approach for studying intertemporal consumption, there are still several long-standing debates regarding the accuracy of both its assumptions and its predictions. As Deaton notes in [5], one of the oldest challenges to the LCH is whether or not empirical data support the prediction that people “smooth” their consumption profiles over the course of their life. For example, it is well-documented that many senior citizens do not run down their assets but instead, continue to save their income such as social security. Furthermore, people do not begin saving when they start earning money. They instead put off saving until their middle-ages, experiencing a sharp decrease in consumption during retirement.

These critiques have resulted in revisions to the LCH since its original formulation in 1954. For example, one reason the elderly do not dissave their assets could be because they are unsure of when they might pass away. For example, an 80 year old man may expect to pass away in the next five years, but due to unexpected advances in medical technology, lives to be 100. If he had run down his assets to zero by the time he is 85, he may spend the next 15 years in poverty. Thus, he would rather save parts of his income and not draw his assets down to zero.

Another criticism of the LCH is in regards to the uncertainty that agents face about future income streams, an issue Modigliani recognized early on. What happens when agents are young and face extreme uncertainty about future wages, careers, and life expectancy? Alternatively, what if the agent’s career path has low job-security such as a politician or NFL player? Modigliani argued that the main effect of uncertainty would be that agents would increase savings as a precautionary mechanism, serving as a buffer for unexpected emergencies or the unforeseeable future. Whether the empirical evidence supports this claim is subject to debate and is currently a topic of discussion among economists.

The most fundamental challenge to the LCH however, and the topic of which the rest of this paper will be devoted to, is the question of whether or not agents actually behave rationally, have the foresight to estimate future incomes and life expectancy, and the consistency to execute previously-made plans. These criticisms, stemming from neuroscientists, psychologists, and behavioral economists, serve as the motivation for the next two chapters.

2 The Behavioral Life-Cycle Hypothesis

In the late 1980s and 1990s, a coalition of economists, psychologists, and more recently neuroscientists, have investigated the behavioral assumptions of neoclassical models such as Modigliani’s LCH. More specifically, they have produced significant experimental and empirical evidence that agents are not time-consistent; people will begin saving for retirement, provided they start tomorrow. Recognizing their time-inconsistent behavior, agents will self-impose constraints limiting their future decision space. This is why we put our alarm clocks on the other side of the room and go shopping when we are not hungry. However, these behaviors are contradictory to basic economic axioms which state that additional choices can only make us better-off, or in other words, additional constraints can only make us worse-off. It seems then that the behavioral underpinnings of Modigliani’s LCH are contradicted by experimental evidence.

Furthermore, growing empirical data suggest that his theory is more prescriptive than descriptive. Richard Thaler, a leading behavioral economist, wrote about the neoclassical LCH:

The anomalous empirical evidence on consumption falls into roughly two categories. First, consumption appears to be excessively sensitive to income. Over the life-cycle, the young and the old appear to consume too little, and the middle-aged consume too much. Also, year-to-year consumption rates are
too highly correlated with income to be consistent with the model. Second, various forms of wealth do not appear to be as close substitutes as the theory would suggest. In particular, households appear to have very low marginal propensities to consume either pension wealth or home equity, compared to other assets. [9].

In light of these discoveries, economists have suggested various revisions to the LCH, leading to a theory more representative of human behavior. Thus, in this chapter, I present David Laibson’s model of intertemporal consumption, as outlined in [8]. The rest of this chapter is as follows, in section 2.1, I outline Laibson’s model of intertemporal consumption. In section 2.2, I present the implications and results of the behavioral LCH. Section 2.3 concludes this chapter.

2.1 The Quasi-Hyperbolic Consumption Model

Let $x$ be a liquid asset and $z$ be an illiquid asset. Since $z$ is an illiquid asset, if the individual chooses to liquidate their holdings of $z$ in time period $t$, they will be paid in time period $t+1$. For simplicity, suppose $z, x$ have the same rate of return $r$.

**Assumption 2.1.** Consumers make decisions in discrete time.

More formally, $t \in \{1, 2, ..., T\}$. Each period $t$ has four subperiods. In the first period, the consumer’s assets $x_{t-1}, z_{t-1}$ yield a return of $R_t = 1 + r_t$. In the second subperiod, the consumer receives labor income $y_t$ and access to their liquid savings $R_t x_{t-1}$. In the third subperiod, the consumer chooses current consumption $c_t \leq y_t + R_t x_{t-1}$. In the last subperiod, the consumer chooses new asset allocations $x_t, z_t$, subject to:

$$y_t + R_t (z_{t-1} + x_{t-1}) - c_t = z_t + x_t, \quad x_t, z_t \geq 0 \quad (2.1)$$

The next assumption addresses the time-inconsistency of agents. Experimental evidence indicates that agents have a high discount rate over short horizons and low discount rates over long horizons. Thus, instead of an exponentially discounted utility as in the LCH, a quasi-hyperbolic discount rate is assumed. See figure 2 for a graph of the exponential, hyperbolic, and quasi-hyperbolic discount functions.

**Assumption 2.2.** Utility has a quasi-hyperbolic discount rate.

Mathematically, utility at time $t$ is expressed as:

$$U_t = E_t \left[ u(c_t) + \beta \sum_{\tau=1}^{T-t} \delta^\tau u(c_{t+\tau}) \right] \quad (2.2)$$

Assumption (2.2) is a fundamental assumption to the results of the Behavioral Life-cycle Hypothesis and captures the agent’s time-inconsistent preferences. At time period $t$, the agent may plan to consume at time $\tau$ some amount $\hat{c}_{t,\tau}$. However, as $t \to \tau$, the agent chooses some other consumption plan $\tilde{c}_{t,\tau} \neq \hat{c}_{t,\tau}$.

With these assumptions, we can clearly see the behavioral differences between Laibson’s and Modigliani’s model. Instead of modeling a perfectly rational, time-consistent agent, Laibson captures the psychological realities of human behavior.
2.2 Results and Implications

Proposition 2.1. Consumption tracks income.

As noted in the introduction of this chapter, empirical evidence suggests that in contradiction to Modigliani’s LCH, household consumption is excessively sensitive to income. Laibson’s model provides an explanation for this comovement: the agent, during time $t-1$, allocates $x_{t-1}$ and $z_{t-1}$ such as to constrain his future self at time $t$. Therefore, for almost all of the agent’s life, he faces a self-imposed liquidity constraint which ensures obedience to the original consumption plan. Furthermore, in equilibrium, $c_t = y_t + R_t x_{t-1}$. Informally, this means that the individual will consume all resources immediately available to him. Since the agent during period $t-1$ only has control over allocation of assets $x, z$, and no power over the $y_t$, we thus see that when $y_t$ is high, $c_t$ will also be high and vice-versa, proving that consumption tracks income.

The next two propositions address the idea that different sources of wealth have different marginal propensities to consume.

Proposition 2.2. The marginal propensity to consume from liquid assets is one.

More formally, let $c_t = c_t(R_t x_{t-1}, R_t z_{t-1})$ be the equilibrium consumption strategy for the agent in time period $t$. Then

$$
\frac{\partial c_t}{\partial (R_t x_{t-1})} = 1
$$

In other words, the agent dedicates all his liquid assets to consumption, and any increase in liquid assets is used entirely for extra consumption.
Proposition 2.3. The marginal propensity to consume from illiquid assets is zero.

More formally, we have that:

$$\frac{\partial c_t}{\partial (R_t z_{t-1})} = 0$$ (2.4)

Proposition 2.3 states that any change in illiquid assets does not affect consumption in period $t$. This is because perturbations to the agent’s illiquid assets do not affect its liquidity constraint and therefore has no effect on $c_t$.

Propositions 2.1-2.3 paint a very different picture from the results in subsection 1.3. Instead of a rational agent planning and then executing an optimal consumption path as in Modigliani’s LCH, Laibson’s agent must always self-constrain his future self from the desire to over-consume. Even though a fully rational and consistent agent would never invest in $z$ and always invest in $x$, given that their rate of return is the same, we instead see that for a dynamically time-inconsistent agent, the optimal thing to do is to invest significant portions of wealth to $z$.

Furthermore, we have provided a theoretical basis for why different sources of wealth have different marginal propensities to consume. Propositions 2.2 and 2.3 state that consumers, knowing that they are time-inconsistent, follow savings rules to ensure that they do not over consume. Consumers follow heuristic rules about consumption: it is okay to spend extra money on consumption when you receive a Christmas bonus, but any increase in home equity wealth should be saved.

2.3 Discussion and Limitations

Laibson, by using experimental findings from the psychology literature, creates a model involving a dynamically inconsistent agent with access to both liquid and illiquid assets. The agent thus chooses his allocations such that he constrains his future self’s decision space. Furthermore, Laibson explains many of the empirical anomalies to Modigliani’s LCH, namely why people have different marginal propensities to consume different assets and why, in contradiction to basic economic axioms, people willingly constrain their future self’s decision space.

However, there are several limitations to Laibson’s model. The first is that the agent will always face a binding self-imposed liquidity constraint. Since $c_t = y_t + R_t x_{t-1}$, and the agent cannot access $z_t$, the model predicts that after consumption, the agent has no liquid funds available, in contradiction to the behavior of real consumers. Laibson addresses this problem by utilizing the same mechanism as Modigliani, namely introducing a precautionary savings motive. Suppose the agent is uncertain about his future income profile, or that there is some unforeseeable event that occurs with some positive probability. Then the agent will not consume all of his liquid assets as preparation in case of some negative shock.

A more fundamental limitation, and the basis of the third chapter of this paper, is that Laibson models the agent as having no self-control. The only mechanism available to ensure obedience to a consumption plan is an external commitment device: illiquid assets. In reality though, consumers do have internal self-control mechanisms such as will-power, that they use to carry out consumption plans. It is thus to this topic that we devote the final chapter to.
3 The Neuroeconomic Life-Cycle Hypothesis

In the previous two chapters, we have seen a progression of behavioral assumptions starting from a rational, consistent agent with exponentially discounted utility to a time-inconsistent agent with no internal commitment mechanisms, or in other words, no self-control. Alberto Bisin and Jess Benhabib address this problem in [3] by approaching the inter-temporal consumption problem from a neuroeconomics perspective. The rest of this chapter is as follows: subsection 3.1 introduces theories of cognitive control and processes, subsection 3.2 outlines the assumptions and construction of the model, subsection 3.3 proves several major results and discusses their implications, and subsection 3.4 concludes.

3.1 A Cognitive Model of Dynamic Choice.

Suppose that agents have two mechanisms of cognitive processes, automatic processes and control processes.

**Definition 3.1.** *Automatic Processes*: A cognitive process based on the learned association of a specific response to a collection of cues. Underlies classical conditioning and Pavlovian responses.

**Definition 3.2.** *Controlled Processes*: A cognitive process based on the activation, maintenance, and updating of active goals in order to influence cognitive procedures. Possibly inhibits automatic responses.

On the theory of cognitive control, Bisin writes:

Cognitive control is the result of differential activations of automatic and controlled processing pathways. An executive function, or supervisory attention system, modulates the activation levels of the different processing pathways, based on the learned representation of expected future rewards. Cognitive control might fail, as controlled processes fail to inhibit automatic reactions, because actively maintaining the representation of a goal is costly, due to the severe biological limitations of the activation capacity of the supervisory attention system of the cortex. [3].

Benhabib and Bisin state that cognitive control is the main mechanism through which agents exhibit self-control. Mathematically, consider an agent at time $\tau = 0$ who must choose how to allocate an exogenous income endowment $w$ to consumption for two time periods, $t > 0$ and $t+1$. Additionally, suppose the agent has some utility function $U(c)$ for consuming $c$ units of the consumption good, and has an exponential discounting rule with discount rate $\beta$. Then the agent solves the following problem:

$$\max_{c_t, c_{t+1}} \beta^t [U(c_t) + \beta U(c_{t+1})]$$

s.t $c_t + c_{t+1} \leq w$

Let $(c^*, w-c^*)$ be the solution which represents the agent’s plan. Furthermore, suppose that when $\tau = t$, the agent is induced by a strong automatic process that makes him prefer $c^t > c^*$. Thus, when $\tau = t$, the agent’s supervisory attention system must choose whether to activate automatic processes or controlled processes. Let $b$ represent the cost of activating and maintaining controlled processing. Then the agent overrides automatic processes, resists the temptation to consume $c^t$, and consumes $c^*$ if and only if

$$U(c^*) - U(c^t) + \beta [U(w - c^*) - U(w - c^t)] > b$$

The left side of equation 3.1 can be described as a measure of the regret the agent faces if he consumes $c^t$ instead of $c^*$. For a detailed representation of the above intertemporal consumption problem, see Figure 3.
3.2 A Neuroeconomics Approach Intertemporal Consumption

Having introduced the theory of cognitive control, we now extend our analysis to how internal commitment mechanisms affect consumption and savings behavior.

Consider an economy time indexed by \( t = 0, 1, \ldots, \infty \). Let \( k_t, c_t, \) and \( a_t \) denote respectively the agent’s wealth, consumption, and productivity at time \( t \). Then the wealth accumulation equation is:

\[
k_{t+1} = a_t k_t - c_t \tag{3.1}
\]

**Assumption 3.1.** The productivity parameter \( a_t \) is independent and identically distributed (i.i.d), takes values in \((0, \infty)\), and has a well-defined mean \( E(a) > 0 \).

Suppose that at any time \( t \), the agents observe a temptation \( z_t \), which generates a distorted temporary preference of the form \( U(z_t c) \) at time \( t \).

**Assumption 3.2.** The temptation \( z_t \) is i.i.d, takes values in \([1, \infty)\), and has mean \( E[z] > 1 \).

Next, we examine the utility function. We want preferences under temptation \( U(z_t c) \) to have a higher marginal utility of consumption than \( U(c_t) \). This is achieved with the following assumption.

![Delayed gratification choice task]

**Figure 3:** Delayed gratification timeline. Figure taken from [3]
Assumption 3.3. $U(c)$ is Constant Elasticity of Substitution (CES)

Mathematically, this means

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma < 1$$  \hspace{1cm} (3.2)

Assuming that the production technology is linear and $U$ is CES, we can thus restrict our attention to linear consumption plans of the form $c_t = \lambda_t a_t k_t$, where $\lambda_t$ is the consumer’s choice variable and can be interpreted as the propensity to consume at time $t$. Equation (3.1) then takes the form:

$$k_{t+1} = (1 - \lambda_t) a_t k_t$$

Furthermore, assume that the agent has cognitive control; he can either invoke automatic processes and potentially over-consume, or invoke controlled processes which are costly but immune to temptations. Decision making arises from the agent choosing between automatic and controlled processes for each time period. If the agent activates automatic processing, then given $a_t$ and $z_t$, the agent has some propensity to consume $\lambda^A_t$, which is increasing with $z_t$. If the agent activates controlled processing, then given $a_t$, the agent has some propensity to consume $\lambda_t$, which is chosen such that consumption is optimally traded off.

Next, Bisin explains his process for deriving the consumption-saving rule:

The controlled processing pathway first computes the future value of the consumption-saving plan, $D(a_{t+1}, k_{t+1}, z_{t+1})$ which depends on the active process at each future time $t + \tau$, given $a_{t+\tau}$ and $z_{t+\tau}$. Temptations will not be inhibited at all future times as it is costly, in terms of activation capacity, to choose a propensity to consume smaller than the one induced by automatic processing responding to temptation. As in the cognitive control and delayed gratification model in the previous section, we assume that the results of the interaction between processing pathways are determined by a supervisory attention system governed by expected rewards. Suppose in particular that the automatic process is only active if the utility loss (or expected future regret) associated with the temptation is smaller than an exogenous activation cost $b(a, k)$, with the following simple functional form: $b(a, k) = b(a_t k_t)^{1-\sigma}$.

From these assumptions, $D(a_t, k_t, z_t)$ is given by:

$$D(a_t, k_t, z_t) = \max \left[ \begin{array}{c}
\max_{\lambda \geq \lambda^A_t} U(\lambda a_t k_t) + \beta E[D(a_{t+1}, (1 - \lambda) a_t k_t, z_{t+1})], \\
\max_{\lambda} U(\lambda a_t k_t) + \beta E[D(a_{t+1}, (1 - \lambda) a_t k_t, z_{t+1})] - b(a_t k_t)^{1-\sigma}
\end{array} \right]$$  \hspace{1cm} (3.3)

Next, given the future value of the consumption plan $D(a_{t+1}, k_{t+1}, z_{t+1})$, the controlled processing pathway computes the propensity to consume $\lambda_t$ which solves:

$$\max_{\lambda} U(\lambda a_t k_t) + \beta E[D(a_{t+1}, (1 - \lambda) a_t k_t, z_{t+1})]$$  \hspace{1cm} (3.4)

Since this mechanism operates via the controlled processing pathway, the resulting propensity to consume $\lambda^E_t(a_t, k_t)$ is independent of $z_t$.

Lastly, as we mentioned in subsection 3.2, the supervisory attention system chooses to activate automatic or controlled processing pathways based on whether the utility loss of overconsuming is greater than the attention cost $b(a, k)$. Mathematically, the expected future regret is given as:
Proof. Differentiating the maximization problem to obtain first-order conditions, we have that
\[
R(a_t, k_t, z_t) = \left[ \max_{\lambda} U(\lambda a_t k_t) + \beta E[D(a_{t+1}, (1 - \lambda)a_t k_t, z_{t+1})] \right] - \left[ \max_{\lambda \geq \lambda^t} U(\lambda a_t k_t) + \beta E[D(a_{t+1}, (1 - \lambda)a_t k_t, z_{t+1})] \right]
\] (3.5)

Therefore, controlled processing is activated if and only if \(R(a_t, k_t, z_t) > b(a_t k_t)^{1-\sigma}\). To summarize, self-control at time \(t\) requires the active maintenance of a consumption-saving rule independent of temptation \(z_t\). Such maintenance is costly and thus controlled processing is only activated if the utility loss from overconsuming is greater than the maintenance cost. We now present the proof and implications of several results from the outlined model.

3.3 Characterization and Results

We first prove two lemmata, both of which are taken from the appendix of [3]. The first gives a closed form solution of the general consumption-saving maximization problem with stochastic temptations. First, let \(\lambda_t\) denote the solution of the following recursive problem:
\[
V(k_t, z_t) = \max_{\lambda} (1 - \sigma)^{-1} (z_t \lambda a k_t)^{1-\sigma} + \beta E[V((1 - \lambda)a k_t, z_{t+1})]
\] (3.6)

Let \(\bar{z}_t = z_t^{(\sigma-1)/\sigma}\), and \(\gamma = \beta^{-1/\sigma}(a^{-(\sigma-1)/\sigma})\).

**Lemma 1.** The solution\(^1\) of the maximization problem given in equation (3.6), \(\lambda_t\), is:
\[
\lambda_t = \frac{1}{1 + \gamma^{-1} \bar{z}_t E[\bar{z}_{t+1}]^{-1} + \gamma^{-1} \bar{z}_t E[\bar{z}_{t+1}]^{-1} E \sum_{s=t+1}^{s} \prod_{t+1}^{s} \gamma^{t-s}}
\] (3.7)

**Proof.** Differentiating the maximization problem to obtain first-order conditions, we have that
\[
z_t(z_t c_t)^{-\sigma} = \beta E V_1(a k_t - c_t, z_{t+1})
\]
V_1(k_t, z_t) = a \beta E V_1(a k_t - c_t, z_{t+1}) = a(z_t c_t)^{-\sigma} z_t,

which implies that
\[
z_t(z_t c_t)^{-\sigma} = a \beta E (z_{t+1} c_{t+1})^{-\sigma} z_{t+1}
\] (3.8)

Letting \(c_t = \lambda_t a k_t\), we can rewrite (3.8) as
\[
z_t^{-1/\sigma}(z_t \lambda_t) = (a \beta)^{-1/\sigma} (E(\lambda_{t+1} a (1 - \lambda_t))(z_{t+1}))^{-1/\sigma}
\]

Let \(\gamma = (\beta^{-1/\sigma} a^{-1/\sigma})\). Solving for \(\lambda_t\), and doing some manipulation, we get
\[
\lambda_t = \frac{\gamma_t \bar{z}_t^{\frac{1-\sigma}{\sigma}} (E(\lambda_{t+1})(z_{t+1})^{\frac{\sigma-1}{\sigma}})}{1 + \gamma \bar{z}_t^{\frac{1-\sigma}{\sigma}} (E(\lambda_{t+1})(z_{t+1})^{\frac{\sigma-1}{\sigma}})}
\]
\[
= \left[ 1 + \gamma z_t^{\frac{\sigma-1}{\sigma}} \gamma^{-1} (E(\lambda_{t+1})(z_{t+1})^{\frac{\sigma-1}{\sigma}})^{-1} \right]^{-1}
\]

\(^1\)The solution may not be unique.

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Let \( \tilde{z}_t = \tilde{z}_t^{(\sigma-1)/\sigma} \). We guess a solution \( \hat{\lambda}_t \) of the form:

\[
\hat{\lambda}_t = \frac{1}{1 + E \sum_{s=t}^{\infty} \tilde{z}_t (\tilde{z}_{s+1})^{-1} \gamma^{t-s-1}}
\]

\[
= \left[ 1 + \gamma^{-1} \tilde{z}_t E(\tilde{z}_{t+1})^{-1} + \gamma^{-1} \tilde{z}_t E(\tilde{z}_{t+1})^{-1} E \sum_{s=t+1}^{\infty} \gamma^{t+1-s-1} \right]^{-1}
\]

If the guess is correct,

\[
E\lambda_{t+1} \tilde{z}_{t+1} = E \left( 1 + E \sum_{s=t+1}^{\infty} \tilde{z}_t (\tilde{z}_{s+1})^{-1} \gamma^{t+1-s-1} \right)^{-1} \tilde{z}_{t+1}
\]

We substitute \( \hat{\lambda}_t \) into \( \lambda_t \) to check:

\[
\lambda_t = \frac{1}{1 + \tilde{z}_t \gamma^{-1} (E \lambda_{t+1} \tilde{z}_{t+1})^{-1}}
\]

Lastly, after some manipulation, we have:

\[
\lambda_t = \frac{1}{1 + \tilde{z}_t \gamma^{-1} E(\tilde{z}_{t+1})^{-1} (1 + E \sum_{s=t+1}^{\infty} \tilde{z}_t (\tilde{z}_{s+1})^{-1} \gamma^{t+1-s-1})}
\]

\[
= \left( 1 + \tilde{z}_t \gamma^{-1} E(\tilde{z}_{t+1})^{-1} + \tilde{z}_t \gamma^{-1} E(\tilde{z}_{t+1})^{-1} E \sum_{s=t+1}^{\infty} \gamma^{t+1-s-1} \right)^{-1}
\]

We thus conclude that the guess \( \hat{\lambda}_t \) was in fact correct. \( \square \)

We next prove the second lemma, which is a crucial component for the proofs of the propositions. Given an exogenous process \( \lambda_t = \lambda(z_t) \), let

\[
\hat{V}_\lambda(k_t) = (1 - \sigma)^{-1} (\lambda(z_t) ak_t)^{1-\sigma} + E \sum_{r=t+1}^{\infty} \beta^{r-1} (\lambda(z_r) ak_r)^{1-\sigma}
\]

\[
= m^{\lambda}_t(k_t)^{1-\sigma},
\]

where

\[
m^{\lambda}_t = (1 - \sigma)^{-1} (\lambda_t a)^{1-\sigma} + ((1 - \sigma)^{-1} (a)^{1-\sigma}) \left[ \beta ((1 - \lambda_t) a)^{1-\sigma} \right] E(\lambda_{t+1})^{1-\sigma}
\]

\[+ \sum_{s=t+2}^{\infty} \left( (1 - \sigma)^{-1} (\lambda_s a)^{1-\sigma} \left[ \prod_{j=t+1}^{s-1} \beta ((1 - \lambda_j) a)^{1-\sigma} \right] \right)
\]

Given \( E m^{\lambda}_{t+1} \), consider the following maximization problem:

\[
\max_{\lambda_t} \frac{(\lambda_t z_t ak_t)^{1-\sigma}}{(1 - \sigma)} + \beta E m^{\lambda}_{t+1} ((1 - \lambda_t) ak_t)^{1-\sigma}
\]

(3.10)
Lemma 2. The solution to equation 3.10, $\lambda_t$ is (i) increasing in $z_t$, and (ii) decreasing in $E m_{t+1}^\lambda$.

Proof. Taking the first-order conditions and solving for $\lambda_t$, we have

$$\lambda_t = \left(1 + ((z_t)^{\sigma-1}(1 - \sigma)^{-1} \beta E m_{t+1}^\lambda)^{1/\sigma}\right)^{-1}$$

Taking partial derivatives, we have:

$$\frac{\partial \lambda_t}{\partial z_t} > 0,$$

$$\frac{\partial \lambda_t}{\partial E m_{t+1}^\lambda} < 0$$

\[\square\]

Proposition 3.1. The value function $D(a_t, k_t, z_t)$ given in Equation (3.3) exists. The marginal propensity associated with controlled processing $\lambda^E(a_t, k_t)$ is in fact a constant, $\lambda^E$. Furthermore, there exists a unique policy function of problem (3.3), which has the following properties:

(i) It is independent of $(a_t, k_t)$. $\lambda(a_t, k_t, z_t) = \lambda(z_t)$;
(ii) There exists a $\bar{\lambda}$ such that

$$\lambda(z_t) = \begin{cases} 
\max_{\lambda^E} \{\lambda^E, \lambda^I(z_t)\} & \text{for } \lambda^I(z_t) \leq \bar{\lambda} \\
\lambda^E & \text{else}
\end{cases} \tag{3.11}$$

Proof. Consider the maximization problem (3.6), equation 3.9 which gives the value of present and future consumption induced by some consumption-saving rule $\lambda(z)$, and the following maximization problem which determines the agent’s behavior in equilibrium as determined by some consumption-saving rule $\lambda^M(z_t)$:

$$\lambda^M(z_t) = \arg \max_{\lambda} (1 - \sigma)^{-1}(z_t\lambda a_t k_t)^{-\sigma} + EV_{\lambda^M(z)}(a_{t+1}, (1 - \lambda)a_t k_t, z_{t+1})$$

We can write both equation 3.6 and equations 3.9 - 3.12 in the form of problem (3.10). Let $E m_{t+1}^*$ and $E m_{t+1}^M$ denote respectively the expected future value of the program evaluated at the solution of each respective equation. Then $E m_{t+1}^M < E m_{t+1}^*$ since by definition, $\lambda^*(z_t)$ maximizes $E m_{t+1}^\lambda$ with respect to $\lambda$. But then, Lemma 2 implies that $\lambda^M(z_t) > \lambda^*$ for any $z_t$.

\[\square\]

Proposition 3.2. There exists a $\bar{z}$ such that

$$\lambda(z_t) = \begin{cases} 
\max_{\lambda^E} \{\lambda^E, \lambda^I(z_t)\} & \text{for } z_t \leq \bar{z} \\
\lambda^E & \text{else}
\end{cases} \tag{3.12}$$

Proof. First, remember that we assume $a_t = a > 0$, and thus can drop the state variable $a_t$. Let the policy function be denoted $\lambda(k_t, z_t)$. Let:

$$\lambda^E(k_t) = \arg \max_{\lambda} U(\lambda a_k) + \beta E [D((1 - \lambda)a_k, z_{t+1})]$$

$$\lambda^I(k_t, z_t) = \max\{\lambda^E(k_t), \lambda^I(k_t, z_t)\}$$
It then follows that $D(k_t, z_t)$ can be written as:

$$
D(k_t, z_t) = \max \left[ \frac{U(\lambda_t^H ak_t)}{\max \lambda U(\lambda ak_t) + \beta E[D((1 - \lambda)ak_t, z_{t+1})]} + \beta E[D((1 - \lambda)ak_t, z_{t+1})] - b(ak_t)^{1-\sigma} \right]
$$

We next prove that the policy function satisfies a cut-off rule. Afterwards, we will show that the cut-off, and hence the policy function, is independent of $k_t$. Lastly, we prove $\lambda^E > \lambda^*.$

To prove that the policy function satisfies a cut-off rule, suppose $U(\lambda ak_t) + \beta E[D((1 - \lambda)ak_t, z_{t+1})]$ is concave with respect to $\lambda$, and fix $k_t$. We then have that

$$
\max_{\lambda} U(\lambda ak_t) + \beta E[D((1 - \lambda)ak_t, z_{t+1})]
$$

has a unique solution $\lambda^E$, independent of $z_t$. It then follows that

$$(1 - \sigma)^{-1}(\lambda^E ak_t)^{1-\sigma} + \beta E[D((1 - \lambda^E)ak_t, z_{t+1})] - b(ak_t)^{1-\sigma}
= (1 - \sigma)^{-1}(\lambda^{E_t})^{1-\sigma} + \beta E[D((1 - \lambda^{E_t})ak_t, z_{t+1})]
$$

is satisfied for a value of $\lambda, \lambda > \lambda^E$. By construction,

$$
\frac{\partial}{\partial \lambda} \left[ (1 - \sigma)^{-1}(\lambda^{E_t})^{1-\sigma} + \beta E[D((1 - \lambda^{E_t})ak_t, z_{t+1})] \right] \leq 0 \text{ at } \lambda = \bar{\lambda}
$$

Thus, $\lambda$ represents the cut-off for given $k_t$. Since $k_t$ is arbitrary, we can construct the cut-off $\bar{\lambda}(k_t)$ of the statement. Next, we prove the concavity of

$$
U(\lambda ak_t) + \beta E[D((1 - \lambda)ak_t, z_{t+1})]
$$

Let $q_t = ak_t$. Choose arbitrary concave functions $h, U : R_+ \times R_+ \to R_+$ where $R_+ = [0, \infty)$. Let $U = (1 - \sigma)^{-1}c^{1-\sigma}, 0 < \sigma < 1$. Let the operator $T$ be defined as follows:

$$(Th)(q; z_t) = \max \left[ \frac{U(\lambda_t^H (z_t)q_t)}{\max_{\lambda} U(\lambda q_t) + \beta E[h((1 - \lambda)q_t, z_{t+1})]} \right]
$$

where

$$(3.13)
$$

To show that $D$ is concave, it suffices to show that $T$ preserves the concavity of $h$. Let $q = vq_t^1 + (1 - v)q_t^2$. From the concavity of $U$ and $h$, it follows that:

$$(Th)(q; z_t) \geq \max \left[ \frac{vU(\lambda_t^H (z_t)q_t^1) + \beta E[h((1 - \lambda_t^H (z_t))q_t^1, z_{t+1})]}{\max \lambda U(\lambda q_t^1) + \beta E[h((1 - \lambda)q_t^1, z_{t+1})] - b(q_t^1)^{1-\sigma}]}
+ (1 - v)\left[ \max \lambda U(\lambda q_t^2) + \beta E[h((1 - \lambda)q_t^2, z_{t+1})] - b(q_t^2)^{1-\sigma} \right]
= \max \left[ \frac{vU(\lambda_t^H (z_t)q_t^1) + \beta E[h((1 - \lambda_t^H (z_t))q_t^1, z_{t+1})]}{\max \lambda U(\lambda q_t^1) + \beta E[h((1 - \lambda)q_t^1, z_{t+1})] - b(q_t^1)^{1-\sigma}]}
+ (1 - v)U(\lambda_t^H (z_t)q_t^2) + \beta E[h((1 - \lambda_t^H (z_t))q_t^2, z_{t+1})]
= \max \left[ \frac{vU(\lambda_t^H (z_t)q_t^1) + \beta E[h((1 - \lambda_t^H (z_t))q_t^1, z_{t+1})]}{\max \lambda U(\lambda q_t^1) + \beta E[h((1 - \lambda)q_t^1, z_{t+1})] - b(q_t^1)^{1-\sigma}]}
+ (1 - v)\left[ \max \lambda U(\lambda q_t^2) + \beta E[h((1 - \lambda)q_t^2, z_{t+1})] - b(q_t^2)^{1-\sigma} \right]
$$

The latter follows from $\max(a+b, c+d) \geq \max(a, c, b, d) = \max(\max(a, c), \max(b, d)) \geq 0$ if $a, bc, d \geq 0$. Therefore,

$$(Th)(q; z_t) \geq [v(Th)(q_t^1; z_t)] + (1 - v)(Th)(q_t^2; z_t)$$
Thus, \((Th)(q_t; z_t)\) is concave. Lastly, we turn to the independence of the policy function from \(k_t\). The cut-off \(\bar{\lambda}(a, k_t)\) solves

\[
\max_{\lambda} U(\lambda ak_t) + \beta E[D(a_{t+1}, a_{t+1}(1 - \lambda)ak_t, z_t)] - b(ak_t)^{1 - \sigma} = U(\lambda ak_t) + \beta E[(1 - \lambda)ak_t, z_t]
\]

in \(\lambda\). Now consider

\[
D(k_t, z_t) = \max \left[ U(\lambda^I ak_t) + \beta E[D((1 - \lambda^I)ak_t, z_{t+1}), \max_{\lambda} U(\lambda ak_t) + \beta E[D((1 - \lambda)ak_t, z_{t+1})] - b(ak_t)^{1 - \sigma} \right]
\]

We then guess the that \(D\) has the functional form \(D(k_t, z_t) = M(z_t)(ak_t)^{1 - \sigma}\). Then:

\[
M(z_t)(ak_t)^{1 - \sigma} = \max \left[ (\lambda^I)^{1 - \sigma} + \beta EM(z_{t+1})((1 - \lambda^I))^{1 - \sigma}, \max_{\lambda}(\lambda)^{1 - \sigma} + \beta EM(z_{t+1})((1 - \lambda))^{1 - \sigma} - b \right] (ak_t)^{1 - \sigma}
\]

\[
M(z_t) = \max \left[ (\lambda^I)^{1 - \sigma} + \beta EM(z_{t+1})((1 - \lambda^I))^{1 - \sigma}, \max_{\lambda}(\lambda)^{1 - \sigma} + \beta EM(z_{t+1})(a(1 - \lambda))^{1 - \sigma} - b \right]
\]

(3.14)

The policy function \(\lambda(z_t)\) associated with equation (3.14) is also the policy function associated with equation (3.3), and thus is independent of \(k_t\). Furthermore, the cutoff is independent of \(k_t : \bar{\lambda}(k_t) = \bar{\lambda}\). Lastly, we prove that \(\lambda^E > \lambda^*\). Since

\[
\lambda^E = \arg \max_{\lambda} (\lambda^{1 - \sigma} + \beta EM(z_{t+1})((1 - \lambda)0^{1 - \sigma})
\]

The first order conditions implies that \(\lambda^E\) decrease when \(EM(z_{t+1})\) increases. Furthermore, \(EM(z_{t+1})\) decreases with \(b\). But since \(\lambda^* = \lambda^E\) when \(b = 0\), we conclude that for \(b > 0\), \(\lambda^E > \lambda^*\).

With Proposition 1 and Proposition 2 proved, we learn that the behavior of an agent with conflicting preferences in Bisin’s cognitive model can be summarized as follows: the agent has a simple consumption-saving plan with associated propensity to consume \(\lambda^E\) independent of temptation \(z_t\). However, since controlled processing is costly for the agent, he sometimes succumbs to temptations and activates automatic processing with associated marginal propensity \(\lambda^I(z_t)\). Automatic processing is activated only if succumbing to temptations does not affect the agent’s original consumption-saving plan too much and has no large permanent effects on wealth accumulation. If temptations do become large enough, then the agent will activate controlled processing and inhibit automatic processing.

### 3.4 Discussion

Bisin and Benhabib build off of Laibson’s models by exploring consumption-savings decisions when agents have internal commitment mechanisms, using the cognitive neuroscience literature as a theoretical foundation. They find that the dynamic consumption-saving behavior of an agent can be characterized as a simple consumption-saving goal characterized by a constant marginal propensity to consume \(\lambda^E\). The agent chooses when to use automatic or controlled processes via a cut-off rule to implement the agent’s optimal consumption-saving plan while allowing small perturbations in some stages when invoking automatic processes are not especially costly.

Bisin and Benhabib note in the conclusion of their paper that the construction of their model is speculative and has not yet been tested against experimental or brain imaging data. However, it can easily be tested to examine its empirical validity. Lastly, their paper serves as an illustrative guide on how neuroeconomics can further motivate future research on economic theories of internal commitment and cognitive choice, and that such areas are likely to be fruitful research topics in the future.
References


