Fractional Derivatives and Fractional Mechanics

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Abstract

This paper provides a basic introduction to fractional calculus, a branch of mathematical analysis that studies the possibility of taking any real power of the differentiation operator. We introduce two different definitions of the fractional derivative, namely the Riemann-Liouville and Caputo forms, and examine some basic properties of each. Later, we discuss fractional mechanics, where the time derivative is replaced with a fractional derivative of order $\alpha$. We then solve some simple fractional differential equations leading up to the fractional damped harmonic oscillator problem.

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1. Basic Fractional Operators

1.1. The Fractional Integral

The first fractional operator we will introduce is the fractional integral, which is a generalization of the $n$-tuple iterated integral to any real order. We start by expressing any $n$th iterated integral as a single integral, using Cauchy’s formula for repeated integration.

**Theorem 1.1** (Cauchy formula for repeated integration). Let $f$ be some continuous function on the interval $[a,b]$. The $n$th repeated integral of $f$ based at $a$,
\[
f^{(-n)}(x) = \int_a^x \int_a^{\sigma_1} \cdots \int_a^{\sigma_{n-1}} f(\sigma_n) \, d\sigma_n \, d\sigma_{n-1} \cdots d\sigma_2 \, d\sigma_1,
\]

is given by single integration:

\[
f^{(-n)}(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) \, dt
\]

**Proof.** A proof is given by induction. Consider the base case \( n = 1 \). We have that \((x-t)^{n-1} = (x-t)^0 = 1\), so

\[
\int_a^x f(\sigma_1) \, d\sigma_1 = \frac{1}{(0!)} \int_a^x (x-t)^0 f(t) \, dt = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) \, dt
\]

and the statement holds for \( n = 1 \). Now suppose the statement holds for some arbitrary \( n \). We will prove it for \( n + 1 \) by switching the order of integration:

\[
f^{(-n+1)}(x) = \int_a^x \int_a^{\sigma_1} \cdots \int_a^{\sigma_n} f(\sigma_{n+1}) \, d\sigma_{n+1} \, d\sigma_n \cdots d\sigma_2 \, d\sigma_1
\]

\[
= \frac{1}{(n-1)!} \int_a^x \int_a^{\sigma_1} (\sigma_1 - t)^{n-1} f(t) \, dt \, d\sigma_1
\]

\[
= \frac{1}{(n-1)!} \int_a^x \int_t^x (\sigma_1 - t)^{n-1} f(t) \, d\sigma_1 \, dt
\]

\[
= \frac{1}{(n)!} \int_a^x ((x-t)^n - (t-t)^n) f(t) \, dt
\]

\[
= \frac{1}{(n)!} \int_a^x (x-t)^n f(t) \, dt
\]

The result follows by induction. [6]

This theorem gives us the framework we need to take an integral of any real degree. First note that \( \Gamma(n) = (n-1)! \) for positive integers \( n \), where \( \Gamma \) is the gamma function. We can define the Riemann-Liouville fractional integral by replacing the power \( n \) in the integrand with some \( \alpha \in \mathbb{R}^+ \), and replacing \( (n-1)! \) with \( \Gamma(\alpha) \).

**Definition 1** (Riemann-Liouville Operator). Let \( f \) be a continuous function, \( \alpha \in \mathbb{R}^+ \), and \( t \in \mathbb{R} \). The fractional integral of order \( \alpha \) is defined as:

\[
J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u) \, du \quad [1]
\]
1.2. Fractional Derivatives

Now that we’ve defined the fractional integral, we can easily define fractional differentiation of any positive real power by combining the standard differential operator with a fractional integral of order between 0 and 1. We just have to choose which operator to apply first. For example, we can define the derivative of order 1.5 of a function \( f(t) \) as either of the following:

\[
D^{1.5} f(t) = D^2 J^{0.5} f(t)
\]

\[
D^{1.5} f(t) = J^{0.5} D^2 f(t)
\]

These two approaches provide the basis for two different definitions of the fractional derivative. The first definition, in which the fractional integral is applied before differentiating, is called the Riemann-Liouville fractional derivative. The second, in which the fractional integral is applied afterwards, is called the Caputo derivative. These two forms of the fractional derivative each behave a bit differently, as we will see. Here are their formal definitions:

**Definition 2.** Pick some \( \alpha \in \mathbb{R}^+ \), let \( n \) be the nearest integer greater than \( \alpha \). The Riemann-Liouville fractional derivative of order \( \alpha \) of a function \( f(t) \) is given by:

\[
D^\alpha f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-u)^{n-\alpha-1} f(u) \, du
\]

**Definition 3.** Pick some \( \alpha \in \mathbb{R}^+ \), let \( n \) be the nearest integer greater than \( \alpha \). The Caputo fractional derivative of order \( \alpha \) of a function \( f(t) \) is given by:

\[
D^\alpha_* f(t) = J^{n-\alpha} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{n-\alpha-1} f^{(n)}(u) \, du
\]

These definitions, taken from a paper by Boyadjiev, Ishteva, and Scherer, [1], can be extended to all real powers. For \( \alpha < 0 \), we use the definition \( D^\alpha f(t) = D^\alpha_* f(t) = J^{-\alpha} f(t) \). For \( \alpha = 0 \), we set \( D^0 f(t) = D^0_* f(t) = f(t) \). This extension unifies integration and differentiation as one operator, sometimes referred to as the differintegral [6]. Since there are many legitimate ways to define the fractional derivative, of which the Caputo and Riemann-Liouville forms are only two, there are many different legitimate ways to define the differintegral.

1.3. Examples and properties

**Lemma 1.2.** Let \( \alpha > 0 \), \( C, K \in \mathbb{R} \), and let \( f \) and \( g \) be functions such that their fractional derivatives and integrals exist. Then

\[
J^\alpha(Cf(t) + Kg(t)) = CJ^\alpha f(t) + KJ^\alpha g(t)
\]

\[
D^\alpha(Cf(t) + Kg(t)) = CD^\alpha f(t) + KD^\alpha g(t)
\]

\[
D^\alpha_* (Cf(t) + Kg(t)) = CD^\alpha_* f(t) + KD^\alpha_* g(t)
\]
Proof. The linearity of these operators follows from the linearity of the integer-order derivatives and integrals by which they are defined.

Since \( \int_0^t (t-u)^{\alpha-1} \, du = \frac{t^\alpha}{\alpha} \), the fractional integral of order \( \alpha \) of 1 is given by:

\[
J^\alpha 1 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \, du = \frac{t^\alpha}{\alpha \Gamma(\alpha)} = \frac{t^\alpha}{\Gamma(\alpha+1)}
\]

The \( n \)th fractional integral of order \( \alpha \) of 1 is then given by:

\[
J^{n\alpha} 1 = \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}
\]

**Theorem 1.3.** The Riemann-Liouville derivative of order \( \alpha > 0 \) with \( n-1 < \alpha < n \) of the power function \( f(t) = t^p \) for \( p \geq 0 \) is given by:

\[
D^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}
\]

Proof. We compute the Riemann-Liouville Derivative of power \( \alpha > 0 \) as:

\[
D^\alpha t^p = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-u)^{n-\alpha-1} u^p \, du
\]

Set \( u = vt \) for \( 0 \leq v \leq 1 \), \( du = tdv \) and we get:

\[
D^\alpha t^p = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^1 ((1-v)t)^{n-\alpha-1} (vt)^p \, t \, dv = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} B(p+1, n-\alpha) t^{n+p-\alpha}
\]

\[
= \frac{1}{\Gamma(n-\alpha)} B(p+1, n-\alpha) \frac{\Gamma(n+p-\alpha+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}
\]

\[
= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)} \frac{\Gamma(n+p-\alpha+1)}{\Gamma(p-\alpha+1)} \frac{\Gamma(n+p-\alpha+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}
\]

\[
= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}
\]

\[\square\]
Note: $B$ is the Beta function $B(x, y) = \int_0^1 v^{x-1}(1-v)^{y-1} \, dv$ for $x > 0, y > 0$ and we used the fact that $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. [1]

This formula allows us to quickly find the fractional derivative of any polynomial, by simply taking fractional derivatives of each term separately. Figure 1 shows several graphs of the Riemann-Liouville fractional derivatives of various orders of the function $f(x) = x$.

We would hope that the fractional derivative of a constant function is always zero, but this is simply not always the case. If we use our formula for $D^\alpha t^p$ with $p = 0$, we get $D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, so $D^\alpha k = \frac{kt^{-\alpha}}{\Gamma(1-\alpha)}$, which only evaluates to 0 if $k = 0$.

![Fractional derivatives of a linear function](image)

Figure 1: Riemann-Liouville Derivatives of a linear function [3]

Taking the Caputo Derivative yields different results:

**Theorem 1.4.** The Caputo derivative of order $\alpha > 0$ with $n - 1 < \alpha < n$ of the power function $f(t) = t^p$ for $p \geq 0$ satisfies:

$$D^\alpha t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} & (p > n-1) \\ 0 & (p \leq n-1) \end{cases}$$

**Proof.** In the second case, $p \leq n-1$, so $\frac{d^n}{dt^n} t^p = 0$ and $D^\alpha t^p = \frac{1}{\Gamma(n-\alpha)} \int_0^t 0 = 0$. 


0. In the first case, we have \( \frac{d^n}{dt^n} t^p = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} t^{p-n} \), so our Caputo Derivative is given by:

\[
D^\alpha_* t^p = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{n-\alpha-1} \frac{\Gamma(p+1)}{\Gamma(p-n+1)} u^{p-n} du
\]

\[
= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} \int_0^1 ((1-v)t)^{n-\alpha-1} (vt)^{p-n} t dv
\]

\[
= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} t^{p-\alpha} \int_0^1 (1-v)^{n-\alpha-1} v^{p-n} dv
\]

\[
= \frac{\Gamma(p+1)B(p-n+1,n-\alpha)}{\Gamma(n-\alpha)\Gamma(p-n+1)} t^{p-\alpha} = \frac{\Gamma(p+1)\Gamma(p-n+1)\Gamma(n-\alpha)}{\Gamma(n-\alpha)\Gamma(p-n+1)\Gamma(p-\alpha+1)} t^{p-\alpha}
\]

\[
= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}
\]

\[\square\]

So \( D^\alpha_* t^p \) and \( D^\alpha t^p \) coincide when \( p > n-1 \), but in all other cases \( D^\alpha_* t^p \) evaluates to 0 and \( D^\alpha t^p \) does not. This formula also gives us that \( D^\alpha_* k = kD^\alpha t^0 = 0 \) \( \forall \alpha > 0 \) for all constants \( k \). So the positive order Caputo fractional derivative of a constant is always zero. This is one way in which Caputo derivatives are considered to be more well-behaved than Riemann-Liouville derivatives [1].

2. Fractional Mechanics

For the sake of simplicity and consistency, from here on out we will use the Caputo derivative of order \( \alpha \) with \( 0 < \alpha < 1 \). With fractional calculus, we are now able to construct classical mechanics with fractional derivatives. Given position \( x(t) \), we introduce the fractional velocity \( v(t) \) and fractional acceleration \( a(t) \) as follows:

\[
v(t) = D^\alpha_* x(t), \quad a(t) = D^\alpha_* v(t)
\]

We will also make reference to the Mittag-Leffler function, \( E_{\alpha,\beta} \), which is a very special function which finds widespread use in the world of fractional calculus. Just as the exponential function naturally arises in solutions of integer order differential equations, the Mittag-Leffler function plays an analogous role in the solution of non-integer order differential equations. The Mittag-Leffler function in its most general form depends on two parameters \( \alpha > 0 \) and \( \beta \). It is defined as:

\[
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}
\]
Alternatively, if the $\beta$ parameter is not needed, the Mittag-Leffler function can be defined as:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}$$ [5]

The exponential function is equal to the Mittag-Leffler function for $\alpha = 1$. Figure 2 shows graphs of the Mittag-Leffler function for various parameters.

![Figure 2: The Mittag-Leffler function, defined with several different parameters.](image)

2.1. Basic Fractional Differential Equations

In fractional mechanics, Newton’s second law of motion becomes $F = ma = mD_\alpha^0 v$, where $m$ is the mass of the body in motion. When the force is constant, the body moves with a constant fractional acceleration of $\frac{F}{m}$. Now consider the vertical motion of a body in a resisting medium in which there exists a resisting force proportional to the fractional velocity, as is sometimes the case with viscoelastic drag in certain types of materials like polymers [4]. We assume the body is projected downward with zero initial velocity ($v(0) = 0$) in a uniform gravitational field. For some constant $k$ denoting the ratio of resistance to fractional velocity, the equation of motion is given by:

$$F = mD_\alpha^0 v = mg - kv$$

Applying a fractional integral of degree $\alpha$ and dividing both sides by $m$ we get:

$$v(t) = g\mathcal{J}_\alpha^0[1] - \frac{k}{m}\mathcal{J}_\alpha^0[v(t)]$$

Multiply both sides by $(-\frac{k}{m})^n\mathcal{J}_\alpha^n$ and sum both sides for $n$ from 0 to $\infty$ to get:
\[
\sum_{n=0}^{\infty} \left( -\frac{k}{m} \right)^n J^{n\alpha} v(t) = g \sum_{n=0}^{\infty} \left( -\frac{k}{m} \right)^{n+1} J^{(n+1)\alpha} v(t) + \sum_{n=0}^{\infty} \left( -\frac{k}{m} \right)^n J^{n\alpha} v(t)
\]

\[\iff \sum_{n=0}^{\infty} \left( -\frac{k}{m} \right)^n J^{n\alpha} v(t) - \sum_{n=0}^{\infty} \left( -\frac{k}{m} \right)^n J^{(n+1)\alpha} v(t) = g \sum_{n=0}^{\infty} \left( -\frac{k}{m} \right)^n J^{(n+1)\alpha}[1]\]

The two sums on the left cancel each other out except for the \(n = 0\) case, giving us:

\[v(t) = g \sum_{n=0}^{\infty} \left( -\frac{k}{m} \right)^n J^{(n+1)\alpha}[1]\]

Recall that \(J^{n\alpha}[1] = \frac{t^n}{\Gamma(n\alpha + 1)}\). Replacing \(n\) with \(n + 1\) and plugging this into our formula we get:

\[v(t) = g \sum_{n=0}^{\infty} \left( -\frac{k}{m} \right)^n \frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha + 1)} = \frac{mg}{k} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(m\alpha)^n} \frac{t^n}{\Gamma(n\alpha + 1)}
\]

\[= \frac{mg}{k} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (m\alpha)^n}{\Gamma(n\alpha + 1)} \left[ 1 - \sum_{n=0}^{\infty} \frac{(-k m \alpha)^n}{\Gamma(n\alpha + 1)} \right]
\]

and so we’ve found a solution for \(v(t)\). This problem was solved in a paper by Min Jung and Won Sang Chung [2].

Now consider the fractional harmonic oscillator problem:

\[m D^{2\alpha}_x x(t) = -mw^2 x(t)\]

with the initial conditions:

\[x(0) = A, \quad (D^{\alpha}_x x)(0) = v_0\]

Applying a fractional integral of order \(\alpha\) to the equation and dividing both sides by \(m\), we get:

\[D^{\alpha}_x x(t) - v_0 = -w^2 J^{\alpha} x(t)\]
Move $v_0$ to the other side and integrate once more and we get:

$$x(t) - A = v_0 J^{\alpha} 1 - w^2 J^{2\alpha} x(t)$$

Now multiply both sides of the equation by $(-w^2)^m J^{2\alpha}$ to get:

$$(-w^2)^m J^{2\alpha} x(t) - (-w^2)^m J^{(m+2)\alpha} x(t) = (-w^2)^m v_0 J^{(2m+1)\alpha} [1] + (-w^2)^m A J^{2\alpha} [1]$$

Now summing both sides for $m$ from 0 to $\infty$ yields:

$$\sum_{m=0}^{\infty} (-w^2)^m J^{2\alpha} x(t) - \sum_{m=0}^{\infty} (-w^2)^m J^{2\alpha} x(t) = v_0 \sum_{m=0}^{\infty} (-w^2)^m J^{(2m+1)\alpha} [1] + A \sum_{m=0}^{\infty} (-w^2)^m J^{2\alpha} [1]$$

$$\iff x(t) + \sum_{m=1}^{\infty} (-w^2)^m J^{2\alpha} x(t) - \sum_{m=1}^{\infty} (-w^2)^m J^{2\alpha} x(t) = v_0 \sum_{m=0}^{\infty} (-w^2)^m J^{(2m+1)\alpha} [1] + A \sum_{m=0}^{\infty} (-w^2)^m J^{2\alpha} [1]$$

$$\iff x(t) = v_0 \sum_{m=0}^{\infty} (-w^2)^m J^{(2m+1)\alpha} [1] + A \sum_{m=0}^{\infty} (-w^2)^m J^{2\alpha} [1]$$

$$= v_0 \sum_{m=0}^{\infty} \frac{(-w^2)^m t^{(2m+1)\alpha}}{\Gamma((2m+1)\alpha + 1)} + A \sum_{m=0}^{\infty} \frac{(-w^2)^m J^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$= A \sum_{m=0}^{\infty} \frac{(-1)^m (wt^\alpha)^{2m}}{\Gamma(2m\alpha + 1)} + \frac{v_0}{w} \sum_{m=0}^{\infty} \frac{(-1)^m (wt^\alpha)^{2m+1}}{\Gamma((2m+1)\alpha + 1)}$$

$$= AC_{\alpha,1}(wt^\alpha) + \frac{v_0}{w} S_{\alpha,1}(wt^\alpha)$$

where $C_{\alpha,1}$ and $S_{\alpha,1}$ are the Mittag-Leffler cosine and sine functions:

$$C_{\alpha,1}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m}}{\Gamma(2m\alpha + 1)}, \quad S_{\alpha,1}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+1}}{\Gamma((2m+1)\alpha + 1)}$$

Mittag-Leffler cosine and sine functions can also be written as:

$$C_{\alpha,1}(x) = \frac{1}{2} \left[ E_{\alpha,1}(ix) + E_{\alpha,1}(-ix) \right], \quad S_{\alpha,1}(x) = \frac{1}{2i} \left[ E_{\alpha,1}(ix) - E_{\alpha,1}(-ix) \right]$$

These formulas are analogous to the formulas for cosine and sine in terms of $e^{ix}$ and $e^{-ix}$. [2]
2.2. The Fractional Damped Oscillator

The damped harmonic oscillator is the simplest model in classical mechanics of motion with dissipation which corresponds to frictional force proportional to velocity. We will consider the fractional damped oscillator defined by:

\[ mD_\alpha^\alpha x(t) = -mw^2x(t) - 2m\gamma D_\alpha^\alpha x(t) \]

with the following initial condition:

\[ x(0) = A, \quad (D_\alpha^\alpha x)(0) = v_0 \]

Now we will solve this fractional differential equation, once again using a series approach. Let us assume the solution is given by:

\[ x(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha} \]

Recall that \( D_\alpha^p t^p = \frac{\Gamma(p + 1)}{\Gamma(p - \alpha + 1)} t^{p-\alpha} \) when \( D_\alpha^p t^p \neq 0 \). Inserting our differential equation into our series representation we get the following recurrence relation for each integer \( n \geq 0 \):

\[ c_{n+2} \frac{\Gamma((n+2)\alpha + 1)}{\Gamma(n\alpha + 1)} + 2\gamma c_{n+1} \frac{\Gamma((n+1)\alpha + 1)}{\Gamma(n\alpha + 1)} + w^2 c_n \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + 1)} = 0 \]

\[ \iff \quad c_{n+2} \Gamma((n+2)\alpha + 1) + 2\gamma c_{n+1} \Gamma((n+1)\alpha + 1) + w^2 c_n \Gamma(n\alpha + 1) = 0 \]

Let \( a_n = \Gamma(n\alpha + 1) \) and we have:

\[ a_{n+2} + 2\gamma a_{n+1} + w^2 a_n = 0 \]

We can represent this recurrence relation with the quadratic equation:

\[ t^2 + 2\gamma t + w^2 = 0 \]

Let \( p, q \), be solutions of the quadratic equation and we have:

\[ 0 = (t-p)(t-q) = t^2 - pt - qt + pq \]

\[ \iff \quad t^2 - pt = q(t-p) \]

which allows us to rewrite our recurrence relation as:

\[ a_{n+2} - pa_{n+1} = q(a_{n+1} - pa_{n}) \]

Now there are three possibilities:
**Case 1: Over-damped oscillator:** In this case $\gamma^2 > w^2$ and by the quadratic formula we have:

$$p = -\gamma + \sqrt{\gamma^2 - w^2}, \quad p = -\gamma - \sqrt{\gamma^2 - w^2}$$

**Case 2: Critically damped oscillator:** In this case $\gamma^2 = w^2$ and we have:

$$p = q = -\gamma$$

**Case 3: Under-damped oscillator:** In this case $\gamma^2 < w^2$ and we have:

$$p = -\gamma + i\sqrt{\gamma^2 - w^2}, \quad p = -\gamma - i\sqrt{\gamma^2 - w^2}$$

Iterating through the recurrence relation yields:

$$a_n - p^n a_0 = (a_1 - pa_0)p^{n-1} \sum_{k=0}^{n-1} \left( \frac{q}{p} \right)^k$$

which gives the following solution:

$$a_n = \begin{cases} 
A_1 p^n - B_1 q^n & (p \neq q) \\
p^n a_0 + (a_1 - pa_0)np^{n-1} & (p = q) 
\end{cases}$$

with

$$A_1 = a_0 + \frac{a_1 - pa_0}{p - q}, \quad B_1 = \frac{a_1 - pa_0}{p - q}$$

So the position has the following form:

$$x(t) = \begin{cases} 
\frac{v_0 - qA}{p - q} E_{\alpha,1}(pt^\alpha) - \frac{v_0 - pA}{p - q} E_{\alpha,1}(qt^\alpha) & (p \neq q) \\
AE_{\alpha,1}(pt^\alpha) + \frac{v_0 - pA}{\alpha} E_{\alpha,\alpha}(pt^\alpha) & (p = q) 
\end{cases}$$

According to Won Sang Chung and Min Jung [2], fractional differential equations expressed in terms of Caputo derivatives require the same boundary conditions as their integer-order counterparts, due to the previously-established fact that the Caputo derivative of a constant is always zero. This makes all of the solutions explored in this paper unique.

**References**


http://www2.maths.ox.ac.uk/chebfun/examples/integro/html/FracCalc.shtml

