A Study Of the Representation of Continued Fractions By the Application of Möbius Transformations

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Abstract

the purpose of this study is to thoroughly explore the article, ”A Geometric Representation of Continued Fractions”, and examine how the authors present the proofs for the main theorems under scrutiny. The article, written by Alan F. Beardon and Ian Short, published in May 2014 by the Mathematical Association of America, is a study on the implications of the Convergence Theorem for continued fractions and the relationship between continued fractions and Fractional Linear Transformations (Möbius Transformations) as mappings on the hyperbolic plane. Through this paper, I will briefly introduce examples real or complex continued fractions in an algebraic point-of-view to approach the article at hand, and then give insight as to how Fractional Linear Transformations can geometrically represent real and complex Continued fractions by chains of horocycles and horospheres in hyperbolic space.

1 Introduction

Before diving into the study of real and complex valued Continued Fractions to give insight to the article [1], it should be noted that the article under scrutiny does not focus on the specifics of hyperbolic geometry and visualization of various mappings. However, for continued fractions that are complex valued, it should be noted that the geometric representation of the coefficients of the corresponding continued fraction can be better understood through hyperbolic geometry, rather than Euclidean geometry. As side reference, [2] is an explanation for the short film [3], on the visual interpretation of Fractional Linear Transformations. The illustrations on the video
gives an elementary but easily understandable explanation for the geometry of various groups of functions using stereographic projection. In addition, the early study[4] by Lester Ford in the 1920s, which the article [1] makes multiple references of, is a study of different spheres that are constructed by their base points that are tangent to the complex plane and their radii, both of which are affected by various conditions for which Ford bases his work upon; the work by Ford should also fall under consideration in order to explain the implications of the article at hand[1], since the authors of [1] make clear that their work is inspired by the mentioned study by Ford. The aims of this study is to not only summarize the main points of the article [1], but to also further explain as to how the authors make use of Fractional Linear Transforms as horocycles and horospheres in the hyperbolic space to give geometric representation of the continued fraction at hand.

2 continued fractions

Using the proofs presented in [1], consider infinite complex continued fractions of the form

\[ K(b_n) = b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{b_4 + \cdots}}}. \]  

(1)

where \( b_1, b_2, \ldots \) are complex numbers that characterize the function above. For instance, the equation for \( K(b_n) \) is said to be an integer continued fraction if each \( b_j \) are all integers, and a real continued fraction if each \( b_j \) are all real. Although it is likely important to note the rationality of the fraction itself since it is widely recognized, even by Ford in [4], that the continued fraction that consists of finitely many terms equates to a rational number, the scope of this paper is on the continued fractions that have infinitely many terms \( b_n \) because the value of the coefficient term itself at \( n = \infty \) becomes an important factor when representing each coefficient geometrically on the hyperbolic plane. Before moving onto how each of these terms in the fraction can be represented geometrically, note that even though I will freely use the value of the term at \( n = \infty \), the convergence/divergence of a given continued fraction does not make certain of the rationality of the fraction itself because it is still possible for an irrational continued fraction to converge to an irrational number.
3 Fractional Linear Transformations

Before I begin this section, I’d like to remind that the visual illustrations of the short clip [3], further explained in [2] gives a clear explanation as to how all of the various combinations of the different types of fractional linear transformations can be represented in a plane using stereographic projection. Quickly reviewing the elementary identities of Möbius Transformations, the report in [2] gives the following equation for Möbius Transformations,

\[ f(z) = \frac{\rho e^{i\theta}z + \alpha}{z + \beta} \] (2)

for appropriate \( \alpha, \beta \in \mathbb{C} \) and \( \rho, \theta \in \mathbb{R} \). Moreover, the mapping by \( f \) can be obtained as the composition,

1. translation by \( \alpha \)
2. inversion
3. dilation by \( \rho \)
4. rotation by \( \theta \)
5. translation by \( \beta \)

I do not intend to explore any deeper on the technical details of Fractional Linear Transformations other than what is already introduced in our textbook, as well as the video [3], so I will simply make references to the proofs that may need to be presented for such material, rather than repeating what is already explained by great detail; the aims of this study is primarily to explore the application of fractional linear transformations as algebraic and geometric representations of continued fractions given by the equation in (1). With that said, I will now examine the relationship between continued fractions, given by the equation in (1), and sequences of fractional linear transformations as presented in [1] by Beardon and Short.

Letting \( b_1, b_2, \ldots \) be the infinite sequence of terms defined by the equation in (1), consider the sequence of Möbius Transformations on the extended complex plane given by, \( t_n(z) = b_n + \frac{1}{z} \), and \( T_n = t_1 \circ t_2 \circ \ldots \circ t_n \), for \( n = 1, 2, \ldots \), where the latter is the sequence of pointwise products of the former sequence up to its j-th term. Thus, by applying this pointwise multiplication an infinite amount of times, we can evaluate the sequence of pointwise products for \( n = \infty \).
which gives
\[ T_n(\infty) = \frac{A_n}{B_n} = K(b_n) = b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{b_4 + \cdots}}}. \] (3)

The fraction given by \( \frac{A_n}{B_n} \) is a representation of continued fractions by the ratio of the two sequences of matrix entries, \( A_0, A_1, A_2, \ldots \) and \( B_0, B_1, B_2, \ldots \), each with leading terms \( A_0 = 1 \) and \( B_0 = 0 \). This representation of continued fractions is widely accepted as the representation of generalized continued fractions which are integer continued fractions that refer to the action of the modular group on the upper half plane \( \mathbb{H} \) defined by \( \mathbb{H} = \{ x + iy : y > 0 \} \). Before I conclude this section, it should be noted that the article in [1] gives a loosely presented proof for the ratio in (2) that is examined in much greater detail by Ford in [4] where Ford classifies certain combinations of fractional linear transformations as 'The Group of Picard' to represent the \( S \)-spheres examined in his paper. Moreover, the authors of [1] suggest that Möbius maps of the form \( z \mapsto b + \frac{1}{z} \) are strictly loxodromic, and there is no disc or half-plane in the extended complex plane that is invariant under a strictly loxodromic map. In other words, maps \( z \mapsto b + \frac{1}{z} \) interchange the upper and lower half-planes of the extended complex plane when representing the maps as groups of isometries of the hyperbolic plane. On the next section I will focus on the study of complex continued fractions that are represented as a chain horospheres in three dimensional hyperbolic space. Please note that a horosphere defined in the space, \( \mathbb{H}^3 = \{ (x, y, t) \in \mathbb{R}^3 : t > 0 \} \), is either a Euclidean sphere in \( \mathbb{R}^3 \) that is tangent to the complex plane or otherwise lies in \( \mathbb{H}^3 \), parallel to the complex plane with the point \( \infty \) attached.
4 complex valued continued fractions and horospheres
in the hyperbolic plane

We now examine how a complex valued continued fraction can be geometrically represented
by a chain of horospheres in the hyperbolic plane that each correspond to a unique radius and
base point. As I briefly introduced in the previous section, the following statement sheds light
on the geometry of a horosphere under certain M"obius transformations.

Lemma 4.1. suppose that $f(z) = \frac{az+b}{cz+d}$, where $ad - bc = 1$. If $c \neq 0$, then $f(\Sigma_0)$ has base
point $\frac{a}{c}$ and Euclidean radius $\frac{1}{|c|}$. If $c = \infty$, then $f(\Sigma_0) = \{z + j|a|^2 : z \in \mathbb{C}\} \cup \{\infty\}$

The proof of this Lemma can be accessed in [1. Section 5], where it follows directly from
the geometry of M"obius Transformations acting upon some image $\Sigma_0$ in $\mathbb{H}^3$. In [1], Beardon
and Short calculates the image of $\Sigma_0$ under a M"obius Transformation by measuring the 'height'
of a point $z + tj \in \mathbb{H}^3$, to inductively determine that the following theorem gives a unique
chain of horospheres $\Sigma_0, \Sigma_1, \ldots$ for every sequence $b_1, b_2, \ldots$ defined in (1):

Theorem 4.2. Given a continued fraction $K(b_n)$ with complex coefficients, the sequence
$\Sigma_0, T_1(\Sigma_0), T_2(\Sigma_0), \ldots$ is a chain of horospheres. Conversely, given a chain of horospheres
$\Sigma_0, \Sigma_1, \Sigma_2, \ldots$ there is a unique continued fraction $K(b_n)$ with $T_n = t_1 \circ t_2 \circ \ldots \circ t_n$, for
$n = 1, 2, \ldots$

Going back to the two sequences of transformations introduced in the last section, notice
that if we write $t_n(z) = \frac{ib_n(z) + i}{iz + 0}$ to satisfy Lemma 4.1, we get that $t_n(\Sigma_0)$ is a horosphere
with base point $b_n$ and Euclidean radius $\frac{1}{2}$. Moreover $t_n(\Sigma_0)$ is tangent to $\Sigma_0$ at the point
t_n(j) = b_n + j$ by the application of the quaternion described in [1. Sections 5, 6]. At this time,
I’d like to make a short reference to the figures 6.1, 6.2, and 6.3 in [1], each of which gives a
slightly variant representation of how Theorem 4.2 can be visualized. Figure 6.1, defined on top
of page 397, illustrates the inductive process that shows two horospheres $\Sigma_0 \perp t_n(\Sigma_0)$ at the
point given by $t_n(j)$. Figure 6.2, directly below figure 6.1, gives the geometric representation
of what the beginnings of a chain of horospheres look like - note that the very 'first' horosphere
is the complex plane itself representing a fixed two dimensional plane. Lastly, figure 6.3 on
page 398 illustrates the representation of horospheres that are defined inductively by the use
of a similar proof used in Ford’s [4]. Notice that although figure 6.3 seems to give a similar
representation of horospheres as 6.1 and 6.2, it appeared to me that the representation of the
horospheres with the point at $\infty$ seemed geometrically incorrect because a point at infinity

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should not appear as a point in the plane (the limit at infinity is an arbitrary thought of a 
number that never ends). Although the figure itself gives a good representation of what the 
proof implies, this argument cannot necessarily guarantee to hold satisfactory for the infinite 
amount of points that it seems to convey, and I do believe that part of the reason for this is that 
the two-dimensional matrix argument is better equipped to prove how and what a continued 
fraction converges to, whereas the theorem that I covered is more towards classifying various 
types of continued fractions as transformations in n-dimensions.

With better understanding on what the illustrations of [1] convey I now move onto showing 
how Beardon and Short present their proof for Theorem 4.2. Letting M"obius Transformations 
t(n)(z) = b_n + \frac{1}{z}, and T_n = t_1 \circ t_2 \circ ... \circ t_n, for n = 1, 2, ..., suppose that \Sigma_0, \Sigma_1, \Sigma_2, ... is a chain of horospheres with base points z_0, z_1, z_2, ... . Following the inductive process used in [1], let 
b_1 = z_1, and b_n = T_{n-1}^{-1}(z_n) corresponding to the sequence of terms b_n as in (1). It follows that 
t_n(\infty) = b_n = T_{n-1}^{-1}(z_n) and T_n(\infty) = z_n, since the condition by z_n \neq z_{n-1} ensures that b_n \neq \infty (refer to [1. Section 3]) because b_n = T_{n-1}^{-1}(z_n) \neq T_{n-1}^{-1}(z_{n-1}) = \infty. Following the convergence theorem for continued fractions in [1. Section 4], it suffices that \Sigma_0, T_1(\Sigma_0), T_2(\Sigma_0), ... is a chain of horospheres with base points z_0, z_1, z_2, ... and the continued fraction defined in (1) is unique by the consequence that b_1 = z_1, and b_n = T_{n-1}^{-1}(z_n)

In my opinion, the value behind this representation of continued fractions as horospheres 
as used by Beardon and Short gives a more precise interpretation of the chain at a geometric 
level than the representation that Ford implements in [4]. Although it is true that the value 
of the modulus given by inductively taking the determinant of the matrix with entries that 
correspond to values of the terms in the fraction \frac{A_n}{B_n} in (1) can easily be set to examine 
the sequences A_0, A_1, A_2, ... and B_0, B_1, B_2, ... at a deeper algebraic level. I conclude this 
section with a quote from the article [1] itself, which states that "The definition of a chain of 
horospheres extends in a straightforward fashion to N-dimensional hyperbolic space defined 
by \mathbb{H}^N = \{(x_1, x_2, ..., x_N : x_N > 0)\}''.

5 Concluding Remarks

By the application of Möbius Transformations that map from one horosphere to another horo-
sphere in the third dimensional hyperbolic space, we saw how each complex valued continued
fractions described by the equation,

\[ K(b_n) = b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{b_4 + \cdots}}}. \] (4)

corresponds to a unique chain of horospheres \( \Sigma_0, T_1(\Sigma_0), T_2(\Sigma_0), \ldots \) depending on the value of its sequence of complex numbers. Of course, I only briefly discussed other forms of continued fractions, and these have a slightly different transformation of their own. The authors of [1] also dive into how the real valued continued fractions discussed in Ford’s [4] can be represented in a subset of the three-dimensional hyperbolic space, called the vertical hyperbolic plane, but this aspect of continued fractions requires a deeper knowledge of hyperbolic geometry than what I can fathom, and even the authors of [1] admit that it is a study yet to be fully explored. The uniqueness of each of these horospheres could have implications on how each of the terms in the sequence can be defined, and this geometric approach could prove to be more applicable in higher dimensions because the recurrence relations of the two-by-two matrices really only makes sense in the complex plane.

6 References

2. D. N. Arnold, J. Rogness, Möbius Transformations Revealed,
3. both short film and article available at http://www.ima.umn.edu/arnold/moebius/