A Review of the Article: Computing the Laurent Series of the Map $\Psi$

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Abstract

This paper will be a review of the article titled "Computing the Laurent Series of the Map $\Psi$" by B. Beilefeld, Y. Fisher, and F. V. Haesler.

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1 Introduction

In the paper "Computing the Laurent Series of the Map $\Psi$" by B. Beilefeld, Y. Fisher, and F. V. Haesler [1], the authors give some important results relating to the Laurent Series for a map for the Mandelbrot Set. The paper first outlines a proof from a previous paper done by Hubbard and Douady [2] that there exists a conformal isomorphism $\Phi$ between the complement of the Mandelbrot Set to the complement of the unit disk in the complex plane. The paper then
uses some of the techniques from Hubbard and Douady’s [2] proof to compute the Laurent series of $\Psi = \Phi^{-1}$. The main result of the paper is that, in the Laurent series of $\Phi$, all of the coefficients are rational with powers of 2 in the denominator and that many of these coefficients are actually zero. This review will go over the parts in these 3 sections that are relevant to the final result.

2 Definitions

Some parts of the paper being analyzed require some knowledge of a few definitions. The definitions needed will be listed here.

2.1 Laurent Series

A Laurent Series expansion of a function $f$ is defined to be:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

2.2 Mandelbrot Set

For $c \in \mathbb{C}$, let $P_c(z) = z^2 + c$. Also, let $P^n_c(z) = P_c(P_c(...P_c(z)...))$ $n$ times. In other words, $P^n_c(z)$ is the $n^{th}$ iterate of $P_c(z)$, where $P^0_c(z) = z$.

Define $A_n(c) = P^n_c(c)$, where $A_0(c) = c$. Thus, we get the following relationship:

$$A_{n+1}(c) = [A_n(c)]^2 + c$$

Let the set $M$ denote the Mandelbrot Set. The Mandelbrot Set is defined as:

$$M = \{c \in \mathbb{C} : |A_n(c)| \not\to \infty, \text{ as } n \to \infty\}$$

2.3 Conformal Isomorphism

A conformal isomorphism is a map that preserves angles and has an inverse.

2.4 Proper Map

A proper map is a continuous function such that inverse images of compact subsets are compact.

2.5 Degree of a Map

A degree of a continuous mapping between two compact oriented manifolds is an integer that represents the number of times the domain manifold wraps around the range manifold.


2.6 \( \text{Res} (\chi (z)) \)

Define \( \text{Res} (\chi (z)) \) to be the coefficient of the \( \frac{1}{z} \) term for a Laurent Series \( \chi (z) \)

3 Overview of Paper

As was said earlier, the paper by B. Beilefeld, Y. Fisher, and F. V. Haesler has three sections. The first section outlines the proof by Hubbard and Douady. After that, the second section of the paper shows how to find the coefficients for the Laurent series for \( \Psi = \Phi^{-1} \). Once the method for computing the coefficients for the Laurent Series of \( \Psi \) has been established, the third section of the paper moves on to prove that, in the Laurent series expansion of \( \Phi \), all of the coefficients are rational with powers of 2 in the denominator and that many of these coefficients are actually zero.

In the first section, the theorem that Hubbard and Douady’s proof proves is that "There exists a conformal isomorphism \( \Phi : \mathbb{C} - M \rightarrow \mathbb{C} - \bar{D} \)." The paper shows this is true by constructing a map \( \Phi : \mathbb{C} - M \rightarrow \mathbb{C} - \bar{D} \) such that \( \Phi \) is an analytic proper map of degree 1.

In the second section, the paper uses some techniques and expressions from the construction of \( \Phi \) to derive an equation which will be used in the computation of the coefficients of the Laurent series for \( \Psi \). It is shown that by substituting a polynomial approximation for \( \Psi \) into the mentioned equation, one can get some of the coefficients. Then it can be shown by induction that one can get as many coefficients as one wants by using this substitution method. During the calculation of these coefficients, it is noted that because of the way the algebra works out, every coefficient ends up being rational with some power of 2 in the denominator.

In the third section, the paper looks at some properties of the Laurent series of \( \Psi \). The main property that is proved is that many of the coefficients for this Laurent series are actually zero. This turns out to be a special case of a theorem that is proved in this section. This theorem states that:

If \( P (x) \) is a polynomial of degree \( d \), then \( P (\Psi (z)) \) has no \( \frac{1}{z^{(2^j + 1)2^n}} \) term when \( d + j \leq 2^n - 2 \).

This paper uses the fact that the derivative of a Laurent series has no \( \frac{1}{z} \) term, which means that the coefficient in front of \( \frac{1}{z} \) is 0. Using some clever substitutions, the fact stated in the previous sentence, and by inducting on \( j \), this paper shows that all of the desired terms in the theorem to be proved end up being the coefficients of the \( \frac{1}{z} \) term for some Laurent series (some series different form the series of \( \Psi \)). This shows that all of these terms are 0, which is what the
theorem states.

This paper also has a fourth and final section that goes over how $\Psi$ isn’t Hölder continuous under certain conditions. I choose to leave that section out of this review as it is not relevant to the main theorem proven in the third section. This fourth section can be found on (Bielefeld [1], page 36).

4 Outline of Hubbard and Douady’s Proof

Theorem 1. There exists a conformal isomorphism $\Phi : C - M \mapsto C - \bar{D}$

Proof. This paper notes that, if $X$ and $Y$ are Riemann surfaces and $f : X \mapsto Y$ is an analytic proper map of degree 1, then $f$ is an isomorphism. In this case, the paper shows this for $\Phi : C - M \mapsto C - \bar{D}$.

For a fixed $c$, this paper finds a map $\varphi_c(z)$ which will conjugate $P_c(z)$ to $z \mapsto z^2$ near infinity in the closure of $C$. In other words, they find a $\varphi_c(z)$ such that the following equation is true:

$$\varphi_c(P_c(z)) = [\varphi_c(z)]^2 \quad (1)$$

for $z$ near $\infty$, and then they show that $\varphi_c(z)$ can be extended to $z = c$ and then they define $\Phi(x) = \varphi_c(z)$.

What the paper wants to find is a $\varphi_c(z)$ with the following property:

$$\varphi_c(z) = \lim_{n \to \infty} [P_c^n(z)]^{2^{-n}} \quad (2)$$

because if this limit was well-defined, it would satisfy equation (1).

Defining $\varphi_c(z)$:

$$T_0 = z$$

$$T_n = \left(1 + \frac{c}{[P_c^{n-1}(z)]^2}\right)\frac{1}{2^n} \quad \text{for } n = 1, 2, 3 \ldots$$

$$\varphi_c(z) = \prod_{i=0}^{\infty} T_i$$
If $S_n = \prod_{i=0}^{n} T_i$, then we can get the following:

$$\begin{align*}
[S_n]^2 &= [T_0 T_1 \ldots T_n]^2 \\
&= z^{2^n} \left[ 1 + \frac{c}{z^2} \right]^{2^{n-1}} \left[ 1 + \frac{c}{(z^2 + c)^2} \right]^{2^{n-2}} \\
&= z^{2^n} \left[ \frac{z^2 + c}{z^2} \right]^{2^{n-1}} \left[ \frac{(z^2 + c)^2 + c}{(z^2 + c)^4} \right]^{2^{n-2}} \\
&= z^{2^n} \left[ \frac{P_1^1 (z)}{z^2} \right]^{2^{n-1}} \left[ \frac{P_2 (z)}{(P_1^1 (z))^2} \right]^{2^{n-2}} \\
&= P_n^c (z) \text{ (telescoping product)}
\end{align*}$$

Thus, we have that $\phi_c (z) = \prod_{i=0}^{\infty} T_i$ which is a well-defined version of equation (2).

The paper then goes on to prove that $\phi_c (z)$ can be defined in a neighborhood of $\infty$ which contains $c$, and that there is a unique way to extend $\phi_c (z)$ analytically to a neighborhood of $\infty$ that contains $c$. The details are a bit technical and so I choose to leave these parts out. Please look at Bielefeld [1] (page 28) for the full proof.

**Defining $\Phi$:**

The map $\Phi : C - M \mapsto C - \bar{D}$ is defined by $\Phi (c) = \phi_c (c)$. $\Phi (C - M) \subseteq C - \bar{D}$. In other words, $|\phi_c (c)| > 1$ because $|\phi_c (c)|^{2^n} = |\phi_c (P_n^c (z))|$ and this goes to infinity for any $c \in C - M$. Also, near infinity, $\Phi$ can be written as $\prod_{i=0}^{\infty} \tau_i$ where

$$\begin{align*}
\tau_0 &= 0 \\
\tau_n &= \left( 1 + \frac{c}{A_{n-1} (c)} \right) \frac{1}{2^n} \text{ for } n \geq 1
\end{align*}$$

and for the $2^{-n}$th root we take the principal branch of the root. The terms converge to 1 quickly enough to ensure that the product converges.

After defining $\Phi$ the paper goes on to show that $\Phi$ is an analytic proper map and that the map has degree 1, thus proving that $\Phi$ is an isomorphism. This concludes the outline for the proof of Douady and Hubbard’s theorem. □

Using this definition of $\Phi (c)$, several estimates can be made on the Laurent series of $\Phi$, which will be used for $\Phi^{-1}$. We have that:

$$\Phi (c) = \lim_{n \to \infty} \Phi_n (c) = \tau_0 \tau_1 \ldots \tau_n$$
\[ \tau_n = 1 + \frac{1}{2^n c^{(2^n - 1)}} + \ldots \]

The paper notes that the Laurent series expansions of \( \Phi(c) \) and \( \Phi_n(c) \) near \( \infty \) have identical terms \( c + a_0 + \frac{a_1}{c} + \frac{a_2}{c^2} + \ldots + \frac{a_k(n)}{c^{k(n)}} \) with \( k(n) = 2^{n+1} - 3 \).

5 Computing the Coefficients of \( \Phi^{-1} \)

The paper first uses \( \Phi_n(c) \) to derive a formula that will be used to compute the coefficients of \( \Phi^{-1} \). What essentially happens is that we calculate the first \( k(n) \) terms of \( \Psi \) by inverting the first \( k(n) \) terms of \( \Phi \). The calculation is as follows:

\[ [\Phi_n(c)]^{2^n} = [\tau_0 \tau_1 \ldots \tau_n]^{2^n} \]
\[ = c^{2^n} \left[ 1 + \frac{1}{c} \right]^{2^n-1} \left[ 1 + \frac{c}{(c^2 + c)^2} \right]^{2^n-2} \ldots \]
\[ = c^{2^n} \left[ \frac{c^2 + c}{c^2} \right]^{2^n-1} \left[ \frac{(c^2 + c)^2 + c}{(c^2 + c)^2} \right]^{2^n-2} \ldots \]
\[ = c^{2^n} \left[ \frac{A_1(c)}{c^2} \right]^{2^n-1} \left[ \frac{A_2(c)}{(A_1(c))^2} \right]^{2^n-2} \ldots \]
\[ = A_n(c) \text{ (telescoping product)} \]
\[ = c^{2^n} + 2^n - 1 c^{2^n-1} + \ldots \]

If \( \Psi = \Phi^{-1} \), then we know that \( \Phi(\Psi(z)) = z \), which implies that \( \Phi_n(\Psi(z)) = z + O\left(\frac{1}{z^{2^n - 1 + k(n)}}\right) \) as \( z \to \infty \). Thus we have that:

\[ [\Phi_n(\Psi(z))]^{2^n} = z^{2^n} + O\left(\frac{1}{2^{(2^n - 1)}}\right) \quad (3) \]
\[ = A_n(\Psi(z)) \quad (4) \]
\[ = \Psi(z)^{2^n} + 2^{n-1} \Psi(z)^{2^{n-1}} + \ldots \quad (5) \]

the rightmost expression, in this case equation (5), is what is primarily used to calculate the coefficients of \( \Psi \).

So we are interested in computing \( \Psi(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots \). We can compute the coefficients \( b_0 \ldots b_n \) by substituting the following approximation, \( \Psi(z) \approx z + b_0 + \frac{b_1}{z} + \ldots + \frac{b_j}{z^j} \), into the right hand side of equation (5). It can be shown by induction that we can use this method to get as many coefficients as we want. Although the paper hints at induction it doesn’t explicitly go through
it (see Bielefeld [1] page 31). For the sake of understandability I will go through the induction process here. A specific example of this process for a specific number of coefficients can be found in the "Base Case" section of the induction proof below, this base case example was taken from page 31 of Bielefeld [1].

**Proposition 1.** The method stated above can be used to calculate as many coefficients as we want.

**Proof.** The proof will use induction on $j$.

**Base Case:**

For the base case we will compute the coefficients $b_0$ and $b_1$. Setting $n = 1$ will be enough to get us these two coefficients. Using a combination of equations (3) - (5) and using the approximation $\Psi(z) \approx z + b_0 + \frac{b_1}{z}$ we can get the following:

$$A_1(\Psi(z)) = z^2 + (2b_0 + 1)z + \left(2b_1 + (b_0)^2 + b_0\right) + \ldots$$

$$= z^2 + O\left(\frac{1}{z}\right)$$

Since the bottom expression has no $z$ or constant term, we get that $2b_0 + 1 = 0$ and $2b_1 + (b_0)^2 + b_0 = 0$. Solving for these two equations gets us $b_0 = -\frac{1}{2}$ and $b_1 = \frac{1}{8}$. This proves the base cases.

**Inductive Step:**

Suppose that we can calculate the coefficients $b_0 \ldots b_{j-1}$, I will show that we will then be able to compute $b_j$. We make the following approximation:

$$\Psi(z) \approx z + b_0 + \frac{b_1}{z} + \ldots + \frac{b_{j-1}}{z^{j-1}} + \frac{b_j}{z^j}$$

(6)

If we plug equation (6) into the right hand side of equation (5) we will get a term of the following form:

$$[2^n b_j + \text{ (terms involving just } b_0, b_1, \ldots, b_{j-1})] z^{2^n - j - 1}$$

By the right hand side of equation (3) we know that for $j \leq k(n)$, the coefficient of $z^{2^n - j - 1}$ is zero. Thus, we get that:

$$2^n b_j + \text{ (terms involving just } b_0, b_1, \ldots, b_{j-1}) = 0$$

(7)

By the inductive hypothesis, we know the coefficients $b_0, b_1, \ldots, b_{j-1}$. Thus, we can solve for $b_j$. This concludes the induction proof. $\square$
It is also important to note that equation (7) in the "Inductive Step" of the
previous proof gives us one of the main results of the paper. Specifically the
result that every coefficient in the Laurent Series of $\Psi$ has a power of 2 in the
denominator. This is because, by solving for $b_j$ in equation (7), we get that for
any arbitrary $j$ (which means for every coefficient):

$$b_j = -(\text{terms involving just } b_0, b_1, \ldots, b_{j-1})$$

6 Properties of the Laurent Series of $\Psi$

The main result that many of the coefficients of the Laurent Series for $\Psi$ are
actually zero, turns out to be special case of the following theorem.

Theorem 2. If $P(x)$ is a polynomial of degree $d$, then $P(\Psi(z))$ has no
\( \frac{1}{z^{(2j+1)2^n}} \) term when $d + j \leq 2^n - 2$

Note that the main result stated above follows from the theorem when
$P(x) = x$.

Several facts need to be noted before a proof of this theorem is attempted.
First of all an observation that will be essential to the proof of Theorem 2 is the
fact that the derivative of a Laurent Series has no \( \frac{1}{z} \) term. From this observation
we get the following claim.

Claim 1. If $P(x)$ and $Q(x)$ are polynomials and $\chi(z) = z + a_0 + \frac{a_1}{z} + \ldots$, then
$Q(\chi(z))\cdot[|P(\chi(z))|^\prime]$ has no \( \frac{1}{z} \) term. In other words, $\text{Res} \ (Q(\chi(z)) \cdot |P(\chi(z))|^\prime) = 0.$

The paper also gives the following claim:

Claim 2. $A_n(\Psi(z))$ satisfies:

$$A_n(\Psi(z)) = z^{2^n} + \frac{B_1(\Psi(z))}{z^{2^n}} + \frac{B_2(\Psi(z))}{z^{3 \cdot 2^n}} + \ldots + \frac{B_j(\Psi(z))}{z^{(2j-1) \cdot 2^n}} + \ldots$$

where

$$B_1(x) = -\frac{x}{2}$$
$$B_2(x) = \frac{1}{2} \left[B_1 - (B_1)^2\right]$$
$$B_3(x) = -B_1 \cdot B_2$$
$$B_4(x) = \frac{1}{2} \left[B_2 - (B_2)^2\right] - B_1 \cdot B_3$$
$$B_5(x) = -B_1 \cdot B_4 - B_2 \cdot B_3$$
and in general we have:

\[
B_{2i} = \frac{1}{2} \left[ B_i - (B_i)^2 \right] - B_1 \cdot B_{2i-1} - B_2 \cdot B_{2i-2} - \ldots - B_{i-1} \cdot B_{i+1}
\]

\[
B_{2i+1} = -B_1 \cdot B_{2i} - B_2 \cdot B_{2i-1} - \ldots - B_i \cdot B_{i+1}
\]

I will leave the proof for this claim out of this review (the original proof can be found on Bielefeld [1] page 34) and instead move on to the proof of Theorem 2.

**Proof of Theorem 2.** If \( P(z) = az^d + \ldots \), we can write

\[
P(\Psi(z)) = az^d + \ldots + \frac{c_0}{z^{2n}} + \ldots + \frac{c_j}{z^{(2j+1)2n}} + \ldots
\]

We want to prove that for all \( c_j \), \( c_j = 0 \). We will do this by induction on \( j \).

**Base Case: \( j=0 \)**

By equations (3) and (4) and Claim 1, we have that

\[
A_n(\Psi(z)) \cdot [P(\Psi)]' = \left[ z^{2n} + O\left( \frac{1}{z^{(2n-1)}} \right) \right] [P(\Psi)]'
\]

has no \( \frac{1}{z} \) term. So when \( d \leq 2^n - 2 \quad (d + j \leq 2^n - 2; j = 0) \), we have that:

\[
0 = \text{Res} \left( z^{2n} + O\left( \frac{1}{z^{(2n-1)}} \right) \right) [P(\Psi)]' = \text{Res} \left( z^{2n} + O\left( \frac{1}{z^{(2n-1)}} \right) \right) \left[ az^d + \ldots + \frac{c_0}{z^{2n}} + \ldots \right]'
\]

\[
= \text{Res} \left( z^{2n} + O\left( \frac{1}{z^{(2n-1)}} \right) \right) \left[ az^d - 1 + \ldots + \frac{-2^n c_0}{z^{2n+1}} + \ldots \right]
\]

\[
= \text{Res} \left( z^{2n} \left[ adz^{d-1} + \ldots + \frac{-2^n c_0}{z^{2n+1}} + \ldots \right] + O\left( \frac{1}{z^{(2n-1)}} \right) \right) \[ \ldots \]
\]

\[
= \text{Res} \left( adz^{2n+d-1} + \ldots + \frac{-2^n c_0}{z} + \ldots \right) + O\left( \frac{1}{z^{(2n-1)}} \right) \[ \ldots \]
\]

\[
= -2^n c_0
\]

This implies that \( c_0 = 0 \), which proves the base case.

**Inductive Step:**

...
Suppose that the theorem holds for \( j \leq k - 1 \). Observe that:

\[
[A_n (\Psi (z))]^{2k+1} = \left[ z^{2^n} + \frac{B_1 (\Psi (z))}{z^{2^n}} + \ldots + \frac{B_k (\Psi (z))}{z^{(2k-1)2^n}} + \ldots \right]^{2k+1}
\]

\[
= \left[ z^{2^n} + \frac{B_1 (\Psi (z))}{z^{2^n}} + \frac{B_2 (\Psi (z))}{z^{3.2^n}} + \ldots + O \left( \frac{1}{z^{(2k+1)2^n - k-1}} \right) \right]^{2k+1}
\]

\[
= z^{(2k+1)2^n} + z^{(2k-1)2^n} Q_1 (\Psi) + z^{(2k-3)2^n} Q_2 (\Psi) + \ldots
\]

\[
\ldots + z^{2^n} Q_k (\Psi) + O \left( \frac{1}{z^{(2^n-k-1)}} \right)
\]

where \( Q_i (\Psi) \) is a polynomial in \( \Psi (z) \) of degree \( i \) which comes from the cross multiplication of all of the \( B_j (\Psi) \).

We now compute the \( \frac{1}{z} \) term of the rightmost expression in the chain of equalities above multiplied by \( [P (\Psi)]' \), which is:

\[
Res \left( z^{(2k+1)2^n} [P (\Psi)]' \right) + Res \left( z^{(2k-1)2^n} Q_1 (\Psi) [P (\Psi)]' \right) + \ldots
\]

\[
\ldots + Res \left( z^{(2k-3)2^n} Q_2 (\Psi) [P (\Psi)]' \right) + \ldots + Res \left( z^{2^n} Q_k (\Psi) [P (\Psi)]' \right) + \ldots
\]

\[
\ldots + Res \left( O \left( \frac{1}{z^{(2^n-k-1)}} \right) [P (\Psi)]' \right)
\]

(8)

So we get that:

\[
Res \left( z^{(2k+1)2^n} [P (\Psi)]' \right) = - (2k+1) \cdot 2^n c_k
\]

Also, for \( i = 1, \ldots, k \) we have that

\[
Res \left( z^{(2(k-1)+1)2^n} Q_i (\Psi) [P (\Psi)]' \right) = 0
\]

(9)

This is because \( Q_i (\Psi) [P (\Psi)]' = [R (\Psi)]' \) where \( R (\Psi) \) is some polynomial of degree \( i + d \), and by the inductive hypothesis \( R (\Psi) \) has no \( \frac{1}{z^{(2(k-1)+1)2^n}} \) term. Note that for each \( i \) in equation (9), the statement holds when \( (i + d) + (k - i) = d + k \leq 2^n - 2 \).

Also, \( Res \left( O \left( \frac{1}{z^{(2^n-k-1)}} \right) [P (\Psi)]' \right) = 0 \) when \( d + k \leq 2^n - 2 \).

Plugging all of these values of 0 into equation (8), we get the following:

\[
0 = Res \left( z^{(2k+1)2^n} [P (\Psi)]' \right) = - (2k+1) \cdot 2^n c_k
\]

This implies that \( c_k = 0 \), which proves the inductive step.

So by induction the main theorem is proved, and this concludes the review of the paper by B. Beilefeld, Y. Fisher, and F. V. Haesler.
References
