maximum stuff

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We have a holomorphic function $f : D \to S = \{u + iv : |u| \leq 1\}$. Let's compose with the map $g(w) = \tan(\frac{\pi w}{4})$ which maps S to D. We use Schwarz on $h(z) = g(f(z))$. So $|h(z)| \leq |z|$ and we get $=$ if and only if $h(z) = cz$, where $|c| = 1$.

Let's do a little translating. $g(w) = \tan(\frac{\pi w}{4}) = \frac{1}{i}\frac{e^{\frac{\pi i w}{2}}-1}{e^{\frac{\pi i w}{2}}+1}$ and in this we substitute $w = f(z)$. $f = g^{-1}(h(z)$ and carry out all the algebra we get

$$f(z) = \frac{2}{\pi i}\log(\frac{1+ih(z)}{1-ih(z)}), \text{ where } |h(z)| \leq |z|.$$

So $|v(z_0)| \leq \frac{2}{\pi}\log(\frac{1+r}{1-r})$ where $r = |z_0|$. And this max of v is achieved when $f(z) = \frac{2}{\pi i}\log(\frac{1+z}{1-z})$, and at $z = r$. Since the real part of f is 0 at this point this is also the value of $|f(r)|$.