Lindelof Maximum Principle

Theorem 1. *Suppose u is bounded and harmonic in a bounded open connected set W. Suppose $\limsup_{z \to p \in \partial W} u(z) \leq M$ except for a countable set $\{q_j \in \partial W\}$. Then $u(z) \leq M$.*

Proof. Choose $\epsilon_j > 0$ so that $\sum_1^\infty \epsilon_j < \infty$. Let d be the diameter of W. Let $a \in W$ and let c be the distance from a to ∂W. Then

$$w(z) = \sum \epsilon_j \log \frac{|z - q_j|}{d}$$

converges uniformly on compact subsets of W to a harmonic function. Hence $v(z) = u(z) + w(z)$ is harmonic on W. Also $v(z) \leq M$. Let $B = \sup\{v(z) : z \in W$. Then there is a point $b \in \overline{W}$ and sequence $z_j \to b,\ z_j \in W$. If v is not constant, $b \in \partial W$. Also $b \neq q_j$ for any j, so it must be that $B \leq M$. Now if we fix a, we see that $v(a) = u(a) + w(a) \leq M$ and since we can choose $\sum \epsilon_j$ to be arbitrarily small, we see that $B \leq M$. ☐

Remark 1. *It's not obvious how to modify this for a set of measure* 0.