# 336 First Midterm Solutions

1. Let $\alpha = a - ib$. Then $\alpha f = au + bv + i(av - bu)$. Since $\Re(\alpha f)$ is constant, $\alpha f$ is constant. Since $\alpha \neq 0$, $\frac{1}{\alpha}\alpha f = f$ is constant.

2. Since $u$ is harmonic on $\mathbb{C}$, $u = \Re(f)$, where $f \in \mathcal{O}(\mathbb{C})$. Let $h = e^f$. Then $|h(z)| = e^u$ and since $u(z) \to 0$ as $z \to \infty$, $|h(z)| \to 1$ as $z \to \infty$. So $h$ is a bounded entire function and by Liouville's theorem, $h$ is constant. Next $0 = h' = f'h$ and since $h(z) \neq 0$, $f' = 0$ (identically). Thus $f$ is constant, so $u$ is constant. Since $u(z) \to 0$ as $z \to \infty$, $u(z) = 0$ for all $z$.

3.

$$\int_{|z|=1} \frac{dz}{(2z-1)(z-2)} = \frac{1}{2}\int_{|z|=1} \frac{dz}{(z-1/2)(z-2)} \tag{1}$$
$$= \frac{1}{2}\int_{|z|=1} \frac{f(z)dz}{(z-1/2)} \tag{2}$$
$$= \frac{1}{2} 2\pi i f(1/2) \tag{3}$$
$$= -\frac{2\pi i}{3}, \tag{4}$$

by Cauchy's integral formula since $f(z) = 1/(z-2)$ is analytic in $|z| \leq 1$.

4. When $z \neq 0$, $f(z) = e^{-z^{-4}}$ is a composition of two analytic functions and hence $f \in \mathcal{O}(\mathbb{C} - 0)$. On the $x$-axis, $y = 0, v = \Im(f) = 0, u = \Re(f) = e^{-x^{-4}}$. Hence $v_x(0,0) = 0$ and

$$u_x(0,0) = \lim_{h\to 0} \frac{e^{-h^{-4}} - 0}{h} \tag{5}$$
$$= \lim_{t\to\infty} \frac{t}{e^{t^4}} \text{ (let } t = 1/x) \tag{6}$$
$$= 0 \text{ by l'Hopital's rule.} \tag{7}$$

Similarly $u_y(0,0) = 0 = v_y(0,0)$. Thus the Cauchy-Riemann equations are satisfied at $(0,0)$. But $f$ is not even continuous at 0. For example, approach 0 along the ray $te^{\frac{i\pi}{4}}$. Then

$$\lim_{t\to 0} e^{t^{-4}} = \infty. \tag{8}$$