# Math 336 $\mathfrak{Final Exam}$, 8:30 am, June 10 , 2013

Name:________________________________________

One notebook-size page of notes is allowed (each side may be used).

1. Prove that $\displaystyle\sum_{1}^{\infty} \frac{\sin nz}{2^n}$ represents an analytic function on $|\Im(z)| < \log 2$.

2. Is there an analytic function $f$ that maps $|z| < 1$ into $|z| < 1$ such that $f(\frac{1}{2}) = \frac{3}{4}, f(\frac{1}{4}) = \frac{1}{3}$?

3. Suppose that $u$ is a harmonic function and $v$ is its conjugate harmonic function. Suppose $u^2 + v^2$ is never 0. Prove that $\log(u^2 + v^2)$ is harmonic.

4. Prove that there is no entire function $f$ with $f(\frac{i}{n}) = \frac{1}{n+1}$ for all $n$. Hint: show that this implies that $f(z) = \frac{z}{z+i}$.

5. (a) Prove that if $\prod(1+z^k)$ converges uniformly on compact subsets of $\{|z|<1\}$.

   (b) Give an example to show that convergence of $\sum a_k$ does not imply convegence of $\prod(1+a_k)$.

6. Let $f$ and $g$ be entire functions so that satisfy $f^2 + g^2 = 1$. Prove that there is an entire function $h$ so that $f = \cos(h),\ g = \sin(h)$.

7. Let $u(x,y),\ v(x,y)$ be continuously differentiable as functions of $(x,y)$ in a domain $\Omega$. Let $f(z) = u(z) + iv(z)$. Suppose that for every $z_0 \in \Omega$ there is an $r_0$ (depending on $z_0$) such that

$$\int_{|z-z_0|=r} f(z)dz = 0,$$

for all $r$ with $r < r_0$. Prove that $f$ is analytic in $\Omega$. Hint: Show that $f$ satisfies the Cauchy-Riemann equations in $\Omega$.

8. Does there exist a function $f$, which is analytic in a neighborhood of 0, for which:

   (a) $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^2}$ for all large integers $n$?
   (b) $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3}$ for all large integers $n$?

   In each case, either give an example or prove no such example exists.