Math 336 Second Midterm Solutions

1. Since

$$\begin{aligned}|(z^7+5z^3+z-2)-5z^3| &= |z^7+z-2| \\ &\le 4 \\ &< 5 = |5z^3| \\ &\le |(z^7+5z^3+z-2)| + |-5z^3|,\end{aligned}$$

on $|z|=1$, z^7+5z^3+z-2 has three roots in $|z|<1$ and no roots on $|z|=1$. Similarly on $|z|=2$,

$$|z^7+5z^3+z-2-z^7| \le 44 < 128 \le |z^7+5z^3+z-2| + |z^7|,$$

so z^7+5z^3+z-2 has seven roots in $|z|<2$ and hence has four roots in $1<|z|<2$.

2. By continuity, since $\lim_{n\to\infty} f(1/n)=0$, $f(0)=0$. Now if f is not identically 0, then $f(z)=z^k g(z)$ where $g(0)\neq 0$. So

$$|f(1/n)| = |\frac{g(1/n)}{n^k}| \le e^{-n},$$

so $|g(1/n)| < n^k/e^n$ and $g(1/n)\to 0$ as $n\to 0$. This implies $g(0)=0$ and is a contradiction. Hence f is identically 0.

3. $f=e^{z-1/z}$ has a Laurent series expansion $\sum_{n=-\infty}^{n=+\infty} a_n z^n$ in $0<|z|<\infty$ and the coefficients are given by the formula

$$a_n = \frac{1}{2\pi i}\int_{|z|=1} e^{z-1/z} z^{-n-1} dz.$$

Subsitute $z=e^{it}$ to get

$$\begin{aligned}a_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{e^{it}-e^{-it}-int} dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(2\sin t - nt)} dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos(2\sin t - nt) + i\sin(2\sin t - nt))\, dt.\end{aligned}$$

Since $\sin(2\sin t - nt$ is odd, its integral is 0. This proves the result.

4. We perform the following manipulations

$$
\begin{aligned}
1+\sin^2 t &= (3-\cos 2t)/2 \\
\int_0^{2\pi} \frac{dt}{1+\sin^2 t} &= 2\int_0^{2\pi} \frac{dt}{3-\cos 2t} \\
&= \int_0^{4\pi} \frac{du}{3-\cos u} \\
&= \frac{4}{i}\int_{|z|=1} \frac{dz}{6z-z^2-1} \\
&= 4i\int_{|z|=1} \frac{dz}{z^2-6z+1} = 4i\int_{|z|=1} \frac{dz}{(z-(3-2\sqrt{2}))(z-(3+2\sqrt{2}))} \\
&= 4i\frac{2\pi i}{(3-2\sqrt{2})-(3+2\sqrt{2})} \\
&= 4i\frac{2\pi i}{-4\sqrt{2}} \\
&= \sqrt{2}\pi.
\end{aligned}
$$

We have use the Cauchy integral formula for the function

$$
\frac{1}{(z-(3+2\sqrt{2}))},
$$

Which is analytic in the disk $|z|<1$.