Entropy and Applications of Cellular Automata

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Abstract

In this paper I will provide an introduction to the theory of cellular automata and review a paper by Michele D'amico, Giovanni Manzini, and Luciano Margara [3], *On computing the entropy of cellular automata*, in which the authors study the topological entropy of cellular automata. The main problems addressed are proving a closed form for the topological entropy of D-dimensional linear cellular automata over $\mathbb{Z}_m$ for $D = 1$ and for $D \geq 2$ and showing how to efficiently compute the entropy of positively expansive cellular automata. I will cover the closed form for the entropy of linear CA.

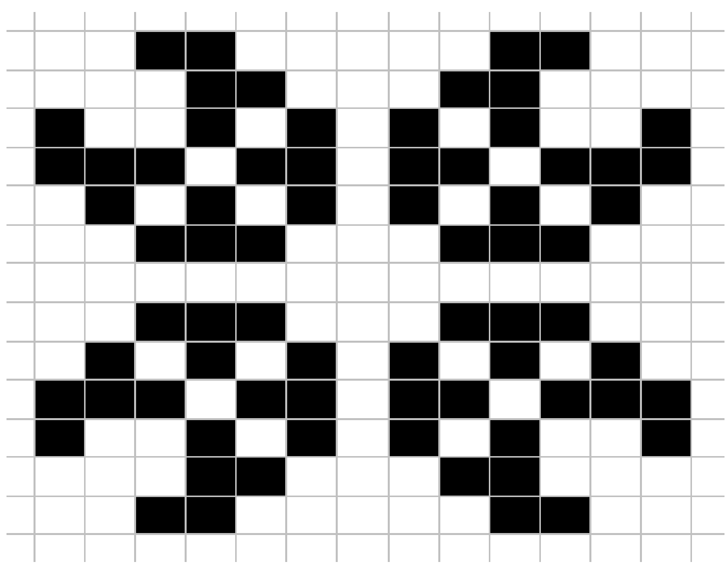

Figure 1: A "pulsar" in Conway's Game of Life.

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1 Introduction

1.1 What are cellular automata?

Cellular automata, abbreviated CA, are a class of discrete models often studied in computability theory, theoretical biology, and physics. A cellular automaton is a regular grid of cells with a finite number of states, for example *black* or *white* as in Figure 1. Then over a discrete number of time steps, t, the state of each cell evolves based on the states of the cells in a neighborhood of itself. The configuration after each time step is called a generation.

The concept was originally discover in the 1940s by Stanislaw Ulam and John von Neumann at Los Alamos National Laboratory. It was not until the 1970s, though, that CA were studied in academia. It was the Game of Life, also known as Life, a cellular automaton devised by British mathematician John Conway in 1970.

1.2 Conway's Game of Life

Conway's Life is a zero-player game, meaning that its evolution is determined by its initial state with no further input. The game is played on an infinite grid of square cells where each cell has two states, *alive* or *dead*. Each cell interacts with the eight cells around it each time step to determine its new state. The rules are as follows:

1. Any live cell with fewer than two live neighbors dies, as if caused by under-population.

2. Any live cell with two or three live neighbors lives on to the next generation.

3. Any live cell with more than three live neighbors dies, as if by over-crowding.

4. Any dead cell with exactly three live neighbors becomes a live cell, as if by reproduction.

The game is particularly interesting because it is a universal Turing machine, that is, anything that can be computed can be computed within the Game of Life. It is this game that opened up the field of cellular automata.

2 Applications

The theory of cellular automata is being used to create ecological, biological, and physical models to great success. In this section I will briefly cover two different practical applications that arose from the study of CA.

2.1 Forest Fires

In [5] a model is presented for predicting the spreading of fire in homogeneous and inhomogeneous forests. The model can easily incorporate weather conditions and land topography. Fires are a part of almost all natural ecosystems, and over the course of many centuries have exerted an exceptionally important influence on the condition of forests all over the planet. The environmental effects of forest fires are huge and there is a constant demand for more effective tools to manage and fight these fires. The authors present the first cellular automata approach to modeling forest fires. They state the problem as follows:

> *Given a scalar velocity field $R(x, y)$ which is the distribution of the rates of fire spread at every point in a forest, the forest fire front at time t_1, the wind direction and speed, and the height and shape of the land, determine the fire front at any time $t_2 > t_1$.*

To create the model the authors partitioned the forest into a uniform grid of square cells. The state is then given by the ratio of burned out cell area to total cell area. So a cell state of 1 would mean that the area around the cell is completely burned out, while a state of 0 would be completely unburned. Then each time step moves the fire a cell length divided by the speed of fire spreading in each direction. This creates a dynamic model for the spread of forest fires which were in good agreement with fire spreading in real forests. This model served as the first basis for the development of algorithms that simulate real fires in real forests.

2.2 Settlement Patterns

In [4] the author applies a cellular automata algorithm to predict the development of a rural countryside near Toronto, Canada. The model was created similarly to the forest fire model in [5], the area was broken up into a grid and environmental factors such as major roads and land topography were used to create the different states that each cell could attain. Two scenarios were run: (1) a static set of rules; and (2) rules that changed as conditions or policies within the township changed. Both models were in agreement with each other and the actual development until the second scenario's rules began to change with the township's own policies. After that point both models still captured and represented selected aspects of the real system. Scenario 1 added new houses mainly on major roads and the density of clusters was quite similar to measured data. While scenario 2 failed to match densities, but more accurately showed the spatial distribution of the clusters.

These are simply two of the early models that came out of the study of cellular automata. This paper will be focused on the results proven by M. D'amico et al. in [3]. The paper focuses on topological entropy, which is one of the most studied properties of dynamical systems. The topological entropy measures the uncertainty of the forward evolution of a dynamical system when a complete description of initial configurations is unknown. Because cellular automata are deterministic dynamical systems, given a complete description of any configuration we may exactly determine the future configurations. Topological entropy gives a quantitative estimation of the uncertainty that is introduced when a complete configuration is not known.

3 Definitions

3.1 Cellular automata

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and natural numbers, respectively. Let $\mathbb{Z}_m$, $m \geq 2$, denote the finite commutative ring of integers modulo m, that is, for $m \geq 2$ let, $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$. We define the **space of configurations**, $\mathscr{C}_m^D = \left\{c \,|\, c : \mathbb{Z}^D \to \mathbb{Z}_m\right\}$, which consists of all functions from $\mathbb{Z}^D$ into $\mathbb{Z}_m$. An element of $\mathscr{C}_m^D$ can be visualized as an infinite D-dimensional lattice in which each cell contains an element of $\mathbb{Z}_m$.

Let $s \geq 1$. A **neighborhood frame** of size s is an ordered set of distinct vectors $\mathbf{u_1}$, $\mathbf{u_2}$, …, $\mathbf{u_s} \in \mathbb{Z}^D$. Given $f : \mathbb{Z}_m^s \to \mathbb{Z}_m$, which we call a **local rule**, a **D-dimensional cellular automata** based on f is defined as the pair $(\mathscr{C}_m^D, F)$, where $F : \mathscr{C}_m^D \to \mathscr{C}_m^D$, is the **global transition map** defined as follows.

$$[F(c)](\mathbf{v}) = f(c(\mathbf{v} + \mathbf{u_1}), \ldots, c(\mathbf{v} + \mathbf{u_s})), \text{ where } c \in \mathscr{C}_m^D,\ \mathbf{v} \in \mathbb{Z}^D. \qquad (1)$$

That is, any cell $\mathbf{v}$ in the configuration $F(c)$ is a function of the content of cells $\mathbf{v} + \mathbf{u_1}$, $\mathbf{v} + \mathbf{u_2}$, …, $\mathbf{v} + \mathbf{u_s}$ in the configuration c. The local rule f and the neighborhood frame completely determine F.

$F(c)$ will denote the result of the application of the map F to the configuration c, and $c(\mathbf{V})$ will denote the value assumed by c in $\mathbf{v}$. For $n \geq 0$, we recursively define $F^n(c)$ by $F^n(c) = F(F^{n-1}(c))$, where $F^0(c) = c$. Let $(\mathscr{C}_m^D, F)$ be a CA based on the local rule f. We say that f is **permutative** in the variable x_i, $-r \leq i \leq r$, if and only if, no matter which values are given to the other $2r$ variables, the modification of the value of x_i causes the modification of the output produced by f. We denote by $f^{(n)}$ the local rule associated to F^n.

3.2 Linear CA over $\mathbb{Z}_m$

In the case of a linear CA the set $\mathbb{Z}_m$ has the usual sum and product operations that make it a commutative ring. We will denote $[x]_m$ to be the integer x taken modulo m. Linear CA have a local rule of the form $f(x_1, \ldots, x_s) = [\sum_{i=1}^{s} \lambda_i x_i]_m$ with $\lambda_1, \ldots, \lambda_s \in \mathbb{Z}_m$. Hence, for a linear D-dimensional CA (1) becomes

$$[F(c)](\mathbf{v}) = \left[\sum_{i=1}^{s} \lambda_i c(\mathbf{v} + \mathbf{u_i})\right]_m, \text{ where } c \in \mathscr{C}_m^D, \mathbf{v} \in \mathbb{Z}^D. \qquad (2)$$

For linear 1-dimensional CA the local rule f can be written as $f(x_{-r}, \ldots, x_r) = [\sum_{i=-r}^{r} a_i x_i]_m$ where at least one between a_{-r} and a_r is nonzero. In this case (1) becomes

$$[F(c)](i) = \left[\sum_{j=-r}^{r} a_j c(i + j)\right]_m, \text{ where } c \in \mathscr{C}_m^1, i \in \mathbb{Z}.$$

3.3 Topological entropy of CA

The topological properties of CA are usually defined with respect to the metric topology induced by the Tychonoff distance over the configuration space $\mathscr{C}_m^D$. With this topology $\mathscr{C}_m^D$ is a Cantor set, that is it is a compact, perfect and totally disconnected set, and every CA is a uniformly continuous map. The definition of topological entropy $\mathscr{H}$ of a continuous map $F : X \to X$ over a compact space X, denoted $\mathscr{H}(X, F)$, is generally accepted as a measure of the complexity of the dynamics of F over X. It is defined in [1], but informally it is a measure of the uncertainty of the evolution of the CA given only partial information on the initial conditions. It is uncomputable for general CA, but in the linear case we will be able to give a closed form, which is the goal of the paper. The entropy of a 1-dimensional CA over $\mathscr{C}_m^1$ satisfies $\mathscr{H}(F) \leq 2r \log m$.

3.4 Lyapunov exponents for CA

The **Lyapunov exponent** of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories and is denoted by λ.

For every $x \in \mathscr{C}_m^1$ and $s \leq 0$ we set

$$W_s^+(x) = \{y \in \mathscr{C}_m^1 : y(i) = x(i) \text{ for all } i \geq s\},$$

$$W_s^-(x) = \{y \in \mathscr{C}_m^1 : y(i) = x(i) \text{ for all } i \leq -s\}.$$

We have that $W_i^+(x) \subset W_{i+1}^+(x)$ and $W_i^-(x) \subset W_{i+1}^-(x)$. For every $n \geq 0$ we define

$$\tilde{\Lambda}_n^+(x) = \min\{s \geq 0 : F^n(W_0^+(x)) \subset W_s^+(F^n(x))\},$$

$$\tilde{\Lambda}_n^-(x) = \min\{s \geq 0 : F^n(W_0^-(x)) \subset W_s^-(F^n(x))\}.$$

Simply put, $\tilde{\Lambda}_n^+(x)$ and $\tilde{\Lambda}_n^-(x)$ have a simple meaning. $W_0^+(x)$ is the set of configurations which agree with x in all cells with index $i \geq 0$. By comparing $F^n(x)$ with $F^n(W_0^+(x))$ the value $\tilde{\Lambda}_n^+(x)$ measures how far differences in cells with index $i < 0$ can propagate to the right-hand side in n iterations of F. Similarly, $\tilde{\Lambda}_n^-(x)$ measures how far differences in cells with index $i > 0$ can propagate to the left-hand side in n iterations of F. We will now introduce an important 1-dimensional CA, the right shift map $(\mathscr{C}_m^1, \sigma)$ defined by $[\sigma(c)](i) = c(i-1)$. Now we can introduce the following shift invariant quantities:

$$\Lambda_n^-(x) = \max_{j \in \mathbb{Z}} \tilde{\Lambda}_n^-(\sigma^j(x)), \quad \Lambda_n^+(x) = \max_{j \in \mathbb{Z}} \tilde{\Lambda}_n^+(\sigma^j(x)), \tag{3}$$

where σ denotes the right shift map. Finally, the values $\lambda^+(x)$ and $\lambda^-(x)$ defined by

$$\lambda^+(x) = \lim_{n \to \infty} \frac{1}{n} \Lambda_n^+(x), \quad \lambda^-(x) = \lim_{n \to \infty} \Lambda_n^-(x) \tag{4}$$

are called, respectively, the right and left Lyapunov exponents of the CA F for the configuration x. If F is linear it is easy to see that these do not depend on x.

3.5 Dynamical properties of CA

For any discrete time dynamical system defined on a metric space, it is possible to define important properties which provide useful information on the long term behavior of the system. On a generic system (X, F), where $F : X \to X$, where we assume that X is equipped with a distance d and that the map F is continuous on X according to the metric topology induced by d. For CA the Tychonoff distance satisfies this property. We denote by $\mathscr{B}(x, \epsilon)$ the open set $\{y \in X : d(x, y) < \epsilon\}$.

A discrete time dynamical system (X, F) is **positively expansive** if and only if there exists $\delta > 0$ such that for every $x, y \in X$, $x \neq y$ there exists $n \geq 0$ such that $d(F^n(x), F^n(y)) > \delta$. The value δ is called the **expansivity constant**.

Intuitively a map is positively expansive if every pair of enough close points eventually separate by at least δ under iteration of F. If a map is positively expansive, then, for all practical purposes, the dynamics of the map defies numerical approximation. Small errors in computation which are introduced by round-off become magnified upon iteration.

A discrete time dynamical system (X, F) is sensitive to initial conditions if and only if there exists $\delta > 0$ such that for any $x \in X$ and for any $\epsilon > 0$, there exists $y \in \mathscr{B}(x, \epsilon)$ and $n \geq 0$, such that $d(F^n(x), F^n(y)) > \delta$. The value δ is called the **sensitivity constant**.

A map is sensitive to initial conditions, or simply sensitive, if there exist points arbitrarily close to x which eventually separate from x by at least δ under iteration of F. Note that not all points near x need eventually separate from x under iteration, but there must be at least one such point in every neighborhood of x.

A discrete time dynamical system (X, F) is **equicontinuous** at $x \in X$ if and only if for any $\delta > 0$ there exists $\epsilon > 0$ such that for any $y \in \mathscr{B}(x, \epsilon)$ and $n \geq 0$ we have $d(F^n(x), F^n(y)) < \delta$.

A discrete time dynamical system (X, F) is equicontinuous if and only if it is equicontinuous at every $x \in X$.

Sensitivity and equicontinuity are related, by comparing the definitions we can easily see that

$$F \text{ is not sensitive} \Leftrightarrow \exists x : F \text{ is equicontinuous at } x. \tag{5}$$

For linear CA we have that $(\mathscr{C}_m^D, F)$ is equicontinuous if and only if it is equicontinuous at a single point $x \in \mathscr{C}_m^D$. So (5) becomes

$$(\mathscr{C}_m^D, F) \text{ is not sensitive } \Leftrightarrow (\mathscr{C}_m^D, F) \text{ is equicontinuous.} \tag{6}$$

4 Computing Entropy - Statement and Proof of Main Theorems

4.1 Theorem 1

Let $(\mathscr{C}_m^1, F)$ be a 1-dimensional CA over $\mathbb{Z}_m$ with local rule $f(x_{-r}, \ldots, x_r) = [\sum_{i=-r}^{r} a_i x_i]_m$, and let $m = p_1^{k_1} \cdots p_h^{k_h}$ be the prime factor decomposition of m. For $i = 1, \ldots, h$ define

$$P_i = \{0\} \cup \{j : \gcd(a_j, p_i) = 1\}, \quad L_i = \min P_i, \quad R_i = \max P_i.$$

Then, the right and left Lyapunov exponents of $(\mathscr{C}_m^1, F)$ are

$$\lambda^+ = -\min_{1 \le i \le h} \{L_i\} \ \text{ and } \ \lambda^- = \max_{1 \le i \le h} \{R_i\}. \tag{7}$$

4.2 Proof of Theorem 1

Let $(\mathscr{C}_m^D, f)$ be a linear CA, and let q be any factor of m. For any configuration $c \in \mathscr{C}_m^D$, $[c]_q$ will denote the configuration in $\mathscr{C}_q^D$ defined by

$$[c]_q(\mathbf{v}) = [c(\mathbf{v})]_q = c(\mathbf{v}) \mod q \text{ for all } \mathbf{v} \in \mathbb{Z}^D.$$

Similarly, F_q will denote the map $[F]_q : \mathscr{C}_q^D \to \mathscr{C}_q^D$ defined by $[F]_q(c) = [F(c)]_q$.

We will need the following lemma, whose proof is provided in [3].

> Let $(\mathscr{C}_{p^k}^1, F)$ be a linear 1-dimensional CA over $\mathbb{Z}_{p^k}$ (p prime) with local rule $f(x_1, \ldots, x_s) = [\sum_{i=1}^{s} a_i x_i]_{p^k}$. Assume there exists a_1 such that $\gcd(a_1, p) = 1$, and let
>
> $$\hat{\mathbf{P}} = \{j : \gcd(a_j, p) = 1\}, \quad \hat{\mathbf{L}} = \min \hat{\mathbf{P}}, \quad \hat{\mathbf{R}} = \max \hat{\mathbf{P}},$$
>
> Then, there exists $h \ge 1$ such that the local rule $f^{(h)}$ associated to F^h has the form
>
> $$f^{(h)}(x_{-hr}, \ldots, x_{hr}) = \left[\sum_{i=h\hat{\mathbf{L}}}^{h\hat{\mathbf{R}}} b_i x_i\right]_{p^k} \text{ with } \gcd(b_{h\hat{\mathbf{L}}}, p) = \gcd(b_{h\hat{\mathbf{R}}}, p) = 1. \tag{8}$$

The proof goes by way of taking the power series of the local rule and then uses induction to show that f^h has the correct form.

The proof will be given only for the left Lyapunov exponent λ^- since the proof for λ^+ is analogous. We know that, since F is a linear map, Lyapunov exponents are independent of the particular configuration considered. Hence, in the rest of the proof we can safely write λ^- and Λ^- istead of $\lambda^-(x)$ and $\Lambda_n^-(x)$.

We first consider the case $m = p^k$ with p prime. From the lemma we know that there exist $h \geq 1$ and $\hat{\mathbf{R}} \in \mathbb{Z}$ such that $f^{(h)}$ is permutative in the variable $x_{h\hat{\mathbf{R}}}$ and does not depend on any other variable x_j with $j > h\hat{\mathbf{R}}$. Let $\lambda_{F^h}^-$ denote the left Lyapunov exponent of the map F^h. If $\hat{\mathbf{R}} \leq 0$ we have that $f^{(h)}$ does not depend on variables with positive index. Hence, if two configurations differ in cells with index $i > \hat{\mathbf{i}}$ such differences never propagate left under iteration of F. We conclude that $\lambda_{F^h}^- = 0$. Assume now that $\hat{\mathbf{R}} > 0$. Let x and x' be two configurations such that $x(i) = x'(i)$ for every $i < \hat{\mathbf{i}}$ and $x(\hat{\mathbf{i}}) \neq x'(\hat{\mathbf{i}})$. Since $f^{(h)}$ is rightmost permutative, we have $[F^h(x)](i) = [F^h(x')](i)$ for every $i < \hat{\mathbf{i}} - h\hat{\mathbf{R}}$ and $x(\hat{\mathbf{i}} - h\hat{\mathbf{R}}) \neq x'(\hat{\mathbf{i}} - h\hat{\mathbf{R}})$, i.e., the difference in cell $\hat{\mathbf{i}}$ moves left exactly $h\hat{\mathbf{R}}$ positions. Hence $\lambda_{F^h}^- = h\hat{\mathbf{R}}$. We now show that $\lambda_{F^h}^- = h\hat{\mathbf{R}}$ implies $\lambda_F^- = \hat{\mathbf{R}}$. From (4) we have

$$\lambda_F^- = \lim_{n\to\infty} \frac{1}{n}\Lambda_n^- = \lim_{n\to\infty} \frac{1}{nh}\Lambda_{nh}^- = \frac{1}{h}\lim_{n\to\infty} \frac{1}{n}\Lambda_{nh}^- = \frac{1}{h}\lambda_{F^h}^-.$$

Since $R = \max(0, \hat{\mathbf{R}})$ then $\lambda_F^- = R$ and the proof is complete.

Consider now the general case $m = pq$, where $\gcd(p,q) = 1$. By the Chinese Remainder Theorem we know that the ring $\mathbb{Z}_m$ is isomorphic to the direct product $\mathbb{Z}_p \otimes \mathbb{Z}_q$. Hence, F^n can be expressed as a linear combination of $[F]_p^n$ and $[F]_q^n$ as follows

$$F^n = \alpha q[F]_p^n + \beta p[F]_q^n$$

where $[\alpha q]_p = 1$ and $[\beta p]_q = 1$. This means that two configurations $F^n(x)$ and $F^n(y)$ differ in the ith cell if and only if the configurations $[F]_p^n([x]_p)$ and $[F]_p^n([y]_p)$ or the configurations $[F]_q^n([x]_p)$ or $[F]_q^n([y]_p)$ differ in the ith cell. Hence, differences in cells with index $i > \hat{\mathbf{i}}$ can propagate left to the farthest cell reachable by either $[F]_p$ or $[F]_q$. Hence, $\lambda_f^- = \max(\lambda_{[F]_p}^-, \lambda_{[F]_q}^-)$. The proof of the theorem follows by a simple inductive argument on the numb of primes in the factorization of the modulus m. □

4.3 Theorem 2

Let $(\mathscr{C}_m^1, F)$ be a 1-dimensional CA over $\mathbb{Z}_m$ with local rule $f(x_{-r}, \ldots, x_r) = [\sum_{i=-r}^{r} a_i x_i]_m$, and let $m = p_1^{k_1} \cdots p_h^{k_h}$ denote the prime factor decomposition of m. Let L_i and R_i be defined as in Theorem 1. Then we can express the topological entropy, $\mathscr{H}$, in the closed form,

$$\mathscr{H}(\mathscr{C}_m^1, F) = \sum_{i=1}^{h} k_i (R_i - L_i) \log(p_i). \tag{9}$$

4.4 Comments on Theorem 2

Theorem 2 follows from a lemma proved by D'amico et al. in [3].

> Let $f(x_{-r}, \ldots, x_r) = [\sum_{i=-r}^{r} a_i x_i]_{p^k}$ be any linear local rule defined over $\mathbb{Z}_{p^k}$ with p prime. Let F be the 1-dimensional global transition map associated to f. Let
>
> $$P = \{0\} \cup \{j : \gcd(a_j, p) = 1\}, \quad L = \min P, \quad \text{and} \quad R = \max P.$$
>
> Then
>
> $$\mathscr{H}(\mathscr{C}_{p^k}^1, F) = k(R - L) \log(p).$$

This lemma is proven using theorem 1 and applying the definition of topological entropy given in [1].

4.5 Putting it all together

Using the previous theorems the authors go on to show that a D-dimensional linear CA over $\mathbb{Z}_m$ with $D \geq 2$ has F sensitive to initial conditions and entropy that is infinite or F is equicontinuous and the entropy is zero. This gives a complete characterization of linear CA over $\mathbb{Z}_m$. These results are the first step in understanding which classes of CA have Lyapunov exponents and topological entropy that are computable. The next step is to attempt to generalize to general CA, not just linear.

5 Further Applications

The more we understand the computability and entropy of different cellular automata the better we can create models which accurately predict the future. This paper is huge step towards solving problems 1 and 2 presented in [8].

Problem 1 is about finding an overall classification of cellular automaton behavior. This approaches that by quantifying the entropy in any linear system. We know that if the entropy is infinite that the system will not yield a homogeneous state and if it is zero that it either evolves to a homogeneous state or simple separated periodic structures. Completing this classification would allow us to further understand cellular automata and the systems which they model.

Problem 2 is about the relationship between entropies and Lyapunov exponents. This paper solves that problem for linear CA in $\mathbb{Z}_m$. This bridges the gap where, intuitively the rate of divergence of trajectories and the entropy of the system ought to be related, but it was previously unproven. This provides hope that we will be able to find a closed form relating the Lyapunov exponents and entropies for different types of CA.

Understanding and being able to compute entropies for cellular automata may lead to the development of CA that can model thermodynamic systems. It is thought that CA may be the key to understanding a large number of subjects on a deeper level. Some wonder if the current model of physics (physics with particle-like objects) could be a cellular automata at its most fundamental level [2].

6 References

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