Mean Value Property

Theorem 1. *Suppose u is a locally integrable function on $\mathbb{C}$. Let r be fixed. Suppose*

$$u(a) = \frac{1}{\pi a^2} \int_{|z-a|\le r} u(z)dA, \tag{1}$$

for all $a \in \mathbb{C}$. Then u is continuous.

Proof. Let $a_j \to a$. Let

$$\chi_j(z) = \begin{cases} 1, & \text{if } |z-a_j| \le r, \\ 0, & \text{if } |z-a_j| > r. \end{cases} \tag{2}$$

$$\chi(z) = \begin{cases} 1, & \text{if } |z-a| \le r, \\ 0, & \text{if } |z-a| > r. \end{cases} \tag{3}$$

$$u(a_j) = \frac{1}{\pi a^2} \int_{|z-a_j|\le r} u(z)dA \tag{4}$$

$$= \frac{1}{\pi a^2} \int_{\mathbb{C}} \chi_j(z)u(z)dA. \tag{5}$$

Then $\chi_j(z)u(z) \to \chi(z)u(z)$ and $|\chi_j u| \le |u|$. By the dominated convergence theorem,

$$u(a_j) = \frac{1}{\pi a^2} \int_{\mathbb{C}} \chi_j(z)u(z)dA \tag{6}$$

$$\to \frac{1}{\pi a^2} \int_{\mathbb{C}} \chi(z)u(z)dA \tag{7}$$

$$= u(a). \tag{8}$$

□

Theorem 2. *Suppose u is locally integrable and there is an r_0 so that*

$$u(a) = \frac{1}{\pi a^2} \int_{|z-a|\le r} u(z)dA, \tag{9}$$

for all $r \le r_0$. Then u is harmonic.

Proof. By the previous theorem, u is continuous. If a continuous function satisfies the mean value property for $r \le r_0$ then when restricted to a disc it assumes its maximum and minimum on the boundary of the disc. Take any disc D and let v be the solution of the Dirichlet problem which equals u on the boundary of the disc. Then the difference satisfies the mean value property in the interior of the disc and hence must assume its maximum and minimum on the boundary. But u and v are equal on the boundary. Hence $u = v$ inside the disc, so u is harmonic. □

Remark 1. *This argument can be modified to show that a locally integrable function that satisfies the mean value property on an open set is harmonic.*