Math 535–Complex Analysis: Problem Set 15

Gill

Due: Friday February 24, 2012 3pm

Problem 1

(a) Show that for $k \geq 2$ the product $\prod_{n=k}^{\infty}(1 + \frac{(-1)^n}{n})$ converges, and find its value.
(b) Determine convergence or divergence of $\prod_{n=1}^{\infty}(1 + \frac{i}{n})$ and of $\prod_{n=1}^{\infty}|1 + \frac{i}{n}|$.

Problem 2

Let D be a proper subdomain of $\mathbb{C}^*$, and $\{z_n\}$ be a seqence of distinct points in D with no limit point in D and $\{m_n\}$ be a sequence of positive integers. Complete the proof started in class that there exists $f \in H(D)$ which has a zero of order m_n at each z_n and no other zeros in D.

Problem 3

Let there be given a sequence $\{z_n\}$ of distinct points in $\mathbb{C}$ with $\lim_{n\to\infty} z_n = \infty$, and sequences $\{a_n\}, \{b_n\}$ in $\mathbb{C}$. Prove that there exists an entire function f such that $f(z_n) = a_n$ and $f'(z_n) = b_n$ for all $n \geq 1$.

Problem 4

Let $\{z_n\}$ be a sequence of not necessarily distinct complex numbers in $\mathbb{D}$ satisfying the Blaschke condition

$$\sum_{n=1}^{\infty}(1 - |z_n|) \leq \infty$$

Prove that there exists a function $f \in H^{\infty}(\mathbb{D})$ such that $f(z_n) = 0$ for $n \geq 1$ and f has no other zeros in $\mathbb{D}$.

Problem 5

Find a constant C such that

$$|\log|e^{i\theta} - 1| \leq \log\frac{\pi}{|\theta|} + C, \qquad \theta \in [-\pi, \pi]$$

Find the best constant if you can.

Problem 6

Let $f \in H(\mathbb{D}(0, R))$, and $0 < r < s < R$.

(a) Prove that

$$\log^+ M(r, f) \leq \frac{s + r}{s - r}T(s, f)$$

(b) Give an example of $f \in H(\mathbb{D})$ for which f is unbounded in $\mathbb{D}$ but $\sup_{0<r<1} T(r, f) < \infty$. Examples like the ones in (b) show, in contrast to (a), there can be no absolute constant for which $\log^+ M(r, f) \leq CT(r, f)$, for every $f \in H(\mathbb{D}(0, R))$ and $0 < r < 1$.

Problem 7 Extra Do Not Turn In

Give an example of a nonconstant bounded holomorphic function in $\mathbb{D}$ such that every point of $\partial\mathbb{D}$ is a limit point of zeros of f.