Math 336 𝔉𝔦𝔫𝔞𝔩𝔈𝔵𝔞𝔪, 8:30 am, June 4 , 2012

Name:____

One notebook-size page of notes is allowed (each side may be used).

1. Suppose that f is an entire function such that

$$|f(z)| \leq a|z|^m + b$$

for all z. Then prove that f is a polynomial of degree at most m.

2. Is there an analytic function f that maps $|z| < 1$ into $|z| < 1$ such that $f(\frac{1}{2}) = \frac{3}{4}, f(\frac{1}{4}) = \frac{1}{3}$?

3. Suppose that u is a harmonic function and v is its conjugate harmonic function. Prove that $u^3 - 3uv^2$ is harmonic.

4. Prove that there is no entire function f with $f(\frac{i}{n}) = \frac{1}{n+1}$ for all n. Hint: show that this implies that $f(z) = \frac{z}{z+i}$.

5. (a) Prove that if $\prod(1 + a_k)$ converges then $\prod |1 + a_k|$ converges.

   (b) Give an example to show that convergence of $\prod |1 + a_k|$ does not imply convegence of $\prod(1 + a_k)$.

6. Find a function, $h(x, y)$, harmonic in $\{y > 0\}$, such that

$$h(x, y) = \begin{cases} 1 & \text{if } 0 < x < 3, y = 0, \\ 0 & \text{if } x > 3, y = 0, \\ 0 & \text{if } x < 0, y = 0 \end{cases}$$

Is this function unique?

7. Let f and g be entire functions so that satisfy $f^2 + g^2 = 1$. Prove that there is an entire function h so that $f = \cos(h),\ g = \sin(h)$.

8. Does there exist a function f, which is analytic in a neighborhood of 0, for which:

   (a) $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^2}$ for all large integers n?

   (b) $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3}$ for all large integers n?

   In each case, either give an example or prove no such example exists.