Stern's Diatomic Sequence

Richard McGovern

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Contents

1 Introduction

The purpose of this paper is to summarize an article by Sam Northshield [1] in which he outlined some particularly beautiful properties of Stern's Diatomic Sequence. This sequence was invented by Moritz Abraham Stern around 1858 and has been the subject of many papers ever since [1]. It is defined by a recurrence relation and has many different interpretations, numeric and geometric.

Among the findings presented by Northshield, it has been shown that the number of ways of expressing an integer n as sums of powers of 2 where each power is used at most twice is equal to the $(n+1)$th term in the Stern's Diatomic Sequence. This sequence has numerous properties closely related to those of the Fibonacci Sequence and indeed, its terms have been shown to be amazing analogues of Fibonacci numbers. This sequence also provides an explicit, one-to-one correspondence of the rationals with the positive integers. Northshield also presents interesting geometrical interpretations of the Diatomic Sequence as infinite hyperbolic graph tilings of the quarter plane in addition to special fractals. These fractals also allow for a visual interpretation of a conjecture that is known to be equivalent to the Riemann Hypothesis.

Perhaps the most significant of all of these results is the relationship between the terms of the Diatomic Sequence, Minkowski's question mark function: ?(x), and its inverse relation. After proper construction of Stern's Diatomic Sequence, and a brief exposure to the other fascinating results, this paper will provide a more detailed explanation of Minkowski's question mark function, its properties, and elaboration of the intimate connections with Stern's Diatomic Sequence. The reader can expect to be particularly interested if he/she is fond of number theory, elegant sequences, or simply loves to revel in the beautiful symmetries that integrate distinct branches of mathematics.

2 Definitions

- **Fibonacci Sequence**

  Let F_n denote the nth Fibonacci number which is recursively defined as: $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ (with $n \geq 2$).

- **Binet Formula**

  An alternate formulation for Fibonacci Numbers:

  $$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} = \frac{\phi^n - \overline{\phi}^n}{\phi - \overline{\phi}}$$

  .

  Where ϕ is the "golden ratio" $\frac{(1+\sqrt{5})}{2}$ and $\overline{\phi}$ is its algebraic conjugate: $\frac{(1-\sqrt{5})}{2}$.

- **Graph**

  A set G(V, E) of vertices $v \in$ V connected by edges $e \in$ E.

  For our purposes, we will only be considering *directed* graphs in which there exists a downward path from vertex v_1 to vertex v_2 but no path from v_2 to vertex v_1.

  To each vertex v_k we can attribute a measure or *degree*. In graphs that we will deal with, the degrees of the vertices will represent terms in a sequence a_k and each term will be finite, though the sequence itself may be unbounded.

  A graph may have any size $|V| + |E|$, including *infinity*, where $|V|$, $|E|$ represent the number of vertices and edges, respectively.

- **Pascal's Triangle**

  A triangular array of numbers with each term equal to the binomial coefficients defined as $\binom{n}{k} = n!/k!(n-k)!$.

3 Stern's Diatomic Sequence

This far-reaching sequence can be defined like so:
Let $a_0 = 0$, $a_1 = 1$, and

$$a_{2n} = a_n$$
$$a_{2n+1} = a_n + a_{n+1}$$

The first few terms include:

$$\{a_k\} = 0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, ...$$

This sequence has been termed "Stern's Diatomic Sequence" which comes from "Stern's Diatomic Array". This is a 2-dimensional array with terms that come from the sequence.

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Figure 1: Stern's Diatomic Array

The first row has the first two terms after zero from Stern's Diatomic Sequence: a_1, a_2.

Then given the nth row, one can generate the elements in the $n+1$st row by copying the elements in the previous row, but between each two entries we insert a new term that is the sum of those two entries. These new terms appear at the top of each new column generated with every new row.

$$
\begin{array}{ccccccccc}
a_1 & & & & & & & & a_2 \\
a_2 & & & & a_3 & & & & a_4 \\
a_4 & & a_5 & & a_6 & & a_7 & & a_8 \\
a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16}
\end{array}
$$

Figure 2: Stern's Diatomic Array in terms of $\{a_n\}$

The diatomic array is also very much like Pascal's Triange in several respects. Each row is palindromic. We can formulate this by considering the $k+1$th element in the $n+1$th row: a_{2^n+k} This is equal to the element with index $k+1$ from the last index in this row: $a_{2^n+k} = a_{2^{n+1}-k}$. Another important result follows: $a_{2^n+k} = a_k + a_{2^n-k}$. Both of these palindromic properties will be used in proofs later.

3.1 A Geometrical Interpretation

One can think of each term in the interior columns as "contributing" to three terms below it, in the following row, and either one or two terms above it in the previous row. Here Northshield defines the *valence* of each term as the number of "contributions" or bonds made with other entries. Since each term makes 3 bonds with entries below it and one or two bonds with entries above it, each term has a *valence* of either 4 or 5. Northshield explains that this is why Stern's Sequence is "Diatomic"–it can be thought of as an alloy of two types of atoms with chemical valence 4 and 5.

For example, consider two such atoms, Carbon and Gold, present in an alloy with a crystal formation. Such a crystal could be described by the Diatomic Array with the right geometrical interpretation. We define an infinite *hyperbolic graph*, $G(V,E) = S_{m,n}$ which is the "boundary" of a tiling of a quarter-plane by $2n$-gons. m is the number of edges that emanate downwards from each vertex v and n is half the number of sides of each polygon tiling the plane. These graphs are "hyperbolic" in the sense that if the tiling was embedded into the hyperbolic upper-half plane, the polygons would have uniform size. [1]

In our case with the theoretical Carbon-Gold crystal, we consider $S_{3,2}$:

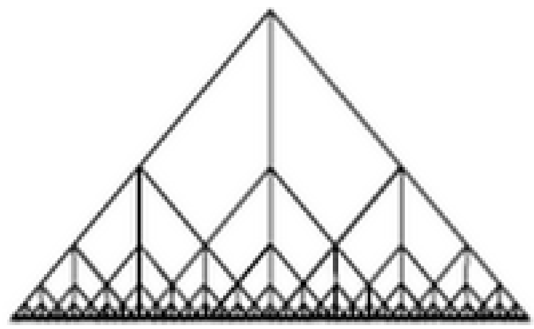

Figure 3: Hyperbolic Graph $S_{3,2}$

If we were to assign a Carbon/Gold atom to each vertex and think of the edges as "bonds" between the atoms, we would get a crystal formation that that could only exist in the hyperbolic upper-half plane where the atoms are stable with a uniform distance apart.

This makes sense when considering some algebraic properties of the Diatomic Array as well. Its row size increases exponentially with n since the number of terms in each row is roughly twice that of the previous row. More precisely, one can show that the nth row of Stern's Diatomic Array contains $2^{n-1}+1$ elements. Also, the sum of the elements in the nth row is $3^{n-1}+1$. [1] Since the row size grows exponentially, the distance between two atoms in the same row would decrease rapidly and we could never have a stable crystal formation.

3.2 The Crushed Array

Notice that the far right column of the diatomic array has the same elements as that on the far left. Northshield notes that removal of this column and squeezing everything to the left yields what he terms the "crushed" array:

1									
1	2								
1	3	2	3						
1	4	3	5	2	5	3	4		
1	5	4	7	3	8	5	7	2	...

Figure 4: Stern's "Crushed" Array

Northshield presents some of its properties as a theorem:

Theorem *The following are properties of the crushed array.*

(a) Each column C_j is an arithmetic sequence and the sequence of differences taken from left to right is:

$$0, 1, 1, 2, 1, 3, 2, 3, 1, 4...$$

Stern's Diatomic Sequence!

(b) The row maxima are 1, 2, 3, 5, 8, 13, ... ; the Fibonacci Sequence!

Proof. Northshield provides a sketch of the proof of these results. (a) is proven by using the fact that each row of the Diatomic Array is a palindrome and obeys the property:

$$a_{2^n+k} + a_k = a_{2^{n+1}+k}.$$

and then reformulates statement (a) as $a_{2^n+k} = na_k + c$ from which it follows that $a_{2^n+k} - na_k = c$ is independent of n.

To prove (b), Northshield notes that every nth row of this crushed array has a maximum M_n at an odd index and the maximum on the following row: $M_{n+1} = a_{2k+1}$ for some k. By definition, this equals $a_k + a_{k+1}$, the sum of two consecutive terms in the previous row–one of which must have an even index, so it must appear in the previous row. Thus,

$$M_{n+1} \leq M_n + M_{n-1}.$$

Since $M_1 = 1 = F_2$ and $M_2 = 2 = F_3$. By induction, $M_n = F_{n+1}$. □

4 A Counting Interpretation

Northshield presented how any integer n can be written as sums of powers of 2 where each power of 2 is used no more than twice. These expressions are called *hyperbinary representations* of n. More precisely,

$$n = e_0 + 2e_1 + 4e_2 + 8e_3 + 16e_4 + ...2^k e_k$$

where $e_i \in \{0, 1, 2\}$. These sums are not unique; in general there are many hyperbinary representations of an integer n. [4]

The following theorem proved by Northshield is rather remarkable:

Theorem *For all n, the number of hyperbinary representations of n equals a_{n+1}, the $(n+1)$th term in Stern's Diatomic Sequence.*

Proof. Northshield proves this using the *generating function* for the Diatomic Sequence:

$$A(x) := \sum_{n=0}^{\infty} a_{n+1} x^n.$$

He then shows that $A(x)$ has a positive radius of convergence by first bounding $a_k \leq F_n \leq \phi^n$, and then arguing: $a_k^{1/k} \leq \phi^{n/2^{n-2}}$ for $k \geq 2^{n-2}$.

For x in the interval of convergence, Northshield shows that

$$A(x) = (1 + x + x^2)A(x^2)$$

.

The continuity of $A(x)$ yields a product:

$$A(x) = (1 + x + x^2)(1 + x^2 + x^4)(1 + x^4 + x^8)(1 + x^8 + x^{16})...$$

The generic term in this product is: $x^{e_0+2e_1+4e_2+8e_3+16e_4+...2^k e_k}$.

Hence, the number of ways we can write n as a hyperbinary representation: $e_0 + 2e_1 + 4e_2 + 8e_3 + 16e_4 + ...2^k e_k$, where $e_i \in \{0, 1, 2\}$, is the coefficient of x^n in $A(x)$ which is a_{n+1} and we are done.

$\square$

There are numerous applications to this theorem, of which this paper has no time to cover.

5 More Connections with the Fibonacci Sequence

We have seen several close connections with the Fibonacci numbers, including from the theorem about the "crushed" array in section 2 that the maximum of the nth row of the diatomic array is F_{n+1}. We shall find that there are many other such connections and in fact the Diatomic numbers are quite good analogues of Fibonacci numbers.

First we return to Pascal's Triangle with boxed terms along one diagonal:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & \boxed{1} & & \\ & 1 & & \boxed{3} & & 3 & & 1 & \\ \boxed{1} & & 4 & & 6 & & 4 & & 1 \end{array}$$

Notice that summing across the diagonal terms yields Fibonacci numbers (in the example above we have $5 = F_5$). More generally,

$$\sum_i \binom{n-i}{i} = \sum_{2i+j=n} \binom{i+j}{i} = F_{n+1}$$

.

Now we create a modified Pascal's Triange: "*Pascal's Triangle Mod 2*" in which each term in Pascal's Triange is replaced with a 1 if it is odd, and a 0 if it is even.

Incredibly, each diagonal sum in Pascal's Triangle Mod 2 is a Diatomic Number:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 0 & & \boxed{1} & & \\ & 1 & & \boxed{1} & & 1 & & 1 & \\ \boxed{1} & & 0 & & 0 & & 0 & & 1 \end{array}$$

We present this as a theorem:

Theorem

$$\sum_{2i+j=n} \binom{i+j}{i} mod 2 = a_{n+1}$$

Proof. The proof of this will be mostly omitted, save for saying that Northshield proves this by way of a lemma which implies $\binom{i+j}{i} mod 2 = 1$ under the condition that i and j share no 1's in the same locations in their base-2 expansions. The result obtained is equivalent to the number of hyperbinary expansions of n, which is a_{n+1}.

For a more detailed explanation, the reader is encouraged to see Allouche and Shallit [5] and Northshield's additional work: [4].

□

6 Enumerating the Rationals

Stern's Diatomic Sequence's great multi-facetedness even extends to a clever enumeration of the rationals. It is well established that the rationals $\mathbb{Q}$ are countable–there exists bijective mappings from all rationals $q \in \mathbb{Q} \rightarrow$ integers $x \in \mathbb{Z}$. Stern's Diatomic Sequence gives one such correspondence quite explicitly:

$$\left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, ..., \frac{a_n}{a_{n+1}}, ...\right\}$$

Where a_j is the jth Diatomic Number.

Every integer appears in this sequence, and no integer appears twice. Northshield presents this as a theorem:

Theorem *The map $n \rightarrow a_n/a_{n+1}$ is a bijection from $\mathbb{Z}^+$ to the set of positive rational numbers.*

Proof. Northshield proves this by showing that every *relatively prime pair* (a, b) appears exactly once in the list defined as:

$$L = [1,1], [1,2], [2,1], [1,3], [3,2], [2,3], [3,1], ..., L_n = [a_n, a_{n+1}], ...$$

Where a *relatively prime pair* is a pair of numbers (a, b) such that their greatest common divisor is 1.

This is done by way of the "Slow Euclidean Algorithm" (SEA) which takes a pair of positive integers, subtracts the smaller from the larger, repeats, and stops when both are equal. [1] Here is an example:

$$[4,7] \rightarrow [4,3] \rightarrow [1,3] \rightarrow [1,2] \rightarrow [1,1]$$

Since this terminated with $[1,1]$ we conclude that (4, 7) is a relatively prime pair. Northshield notes that this algorithm preserves the greatest common divisor at each steps, and in fact always terminates with $[g, g]$, where g is the gcd of some pair of positive integers (a, b).

Now we let $L_n := [a_n, a_{n+1}]$ where a_j is the jth term of the Diatomic Sequence. By the definition of the sequence, for $n > 1$, SEA applied to L_{2n} and to L_{2n+1} both yield L_n. Furthermore, if SEA : $[a, b] \rightarrow L_n$ then either$[a, b] = L_{2n}$ or $[a, b] = L_{2n+1}$. Since $L_1 = [1,1]$, Northshield makes it clear that every L_n is a relatively prime pair.

Now we suppose that there is a relatively prime pair $[a, b]$ that does not appear in our list $\{L_n\}$. If this is so, then all the pairs acquired by applying SEA to $[a, b]$ are not in $\{L_n\}$ either–including $[1,1]$, which is a contradiction. Therefore, every relatively prime pair appears in $\{L_n\}$.

Finally, we prove that each prime pair appears only once in $\{L_n\}$ by supposing not: let $L_n = L_m$ for some $m > n$. If we apply one iteration of SEA to both L_n and L_m, then Northshield argues that this implies $\lfloor n/2 \rfloor = \lfloor m/2 \rfloor$.

This means $m = n+1$. Hence, $a_n = a_{n+1} = a_{n+2}$ which is also a contradiction. [1]

$\square$

From this we can conclude that the function: $f(n) = a_n/a_{n+1}$ maps positive integers $n \in \mathbb{Z}^+$ to positive rationals defined by the ratio of subsequent terms in Stern's Diatomic Sequence. The image of this mapping is the set of all positive rationals, $\mathbb{Q}^+$, and this mapping is in fact bijective.

We conclude this paper with an examination of connections of Stern's Diatomic Sequence with Minkowski's ? (Question mark) function.

7 Minkowski's ? Function

7.1 Continued Fractions

Before diving into possibly the most important result of Northshield's paper, we define notation for a *continued fraction*:

For some integer c_0 and a sequence of positive integers $\{c_k\} = c_1, c_2, c_3, ...$ we define the continued fraction x as

$$x = [c_0; c_1, c_2, c_3, ...] := c_0 + 1/(c_1 + 1/(c_2 + 1/(c_3 + ...))) = c_0 + \cfrac{1}{c_1 + \frac{1}{c_2 + \frac{1}{c_3 + ...}}}$$

Any irrational number can be written in this form given an infinite sequence $\{c_k\}$. In fact, every irrational number $\zeta \in (0,1)$ can be written uniquely in this form [1]. The "golden ratio" ϕ is a well-known example:

$$\phi = [1; 1, 1, 1, ...] = 1 + \cfrac{1}{1 + \frac{1}{1 + \frac{1}{1 + ...}}} = 1 + \frac{1}{\phi}$$

This reveals that the quadratic: $x^2 - x - 1$ has roots: ϕ and $\overline{\phi}$. Notice that the sequence $\{c_k\}$ of terms in the continued fraction expansion for ϕ is periodic (every term is 1). More generally, we have by a theorem [2] that the sequence $\{c_k\}$ associated with the continued fraction for x is periodic if and only if x is the root of a polynomial with degree 2. [1]

This result will be used shortly as we explore Minkowski's enigmatic ? function.

7.2 The ? function

In 1904, Minkowski introduced his ? (Question mark) function, also known as the *Devil's Slippery Staircase*. It is a continuous function with derivative 0 almost everywhere: $? : [0,1] \to [0,1]$. This function takes points $x \in [0,1]$, where x is defined by its continued fraction expansion: $x = [0; c_1, c_2, ...]$, and maps them to points $y \in [0,1]$.

$$?(x) := \sum_k \frac{(-1)^{k-1}}{2^{c_1+c_2+\ldots+c_k-1}} = a_{n+1}$$

Where each term in the power, c_k, is in the sequence of terms definining the continued fraction of x.

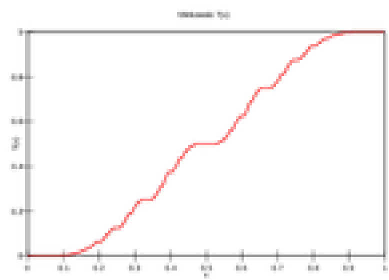

Figure 5: The graph of y = ?(x)

An intriguing feature of this function, as noted by Northshield, is that it maps *quadratic surds* (irrational roots of a quadratic equation with integer coefficients) to *rational numbers.*

Now we define a set $D = \{k/2^n : k, n \in \mathbb{Z}^+, k \leq 2^n\}$ of positive *dyadic rationals* (a rational number whose denominator is a power of 2) in the unit interval, and with this we associate a function $f : D \to \mathbb{Q}^+$ defined by

$$f(k/2^n) := \frac{a_k}{a_{2^n+k}}$$

Again, where every a_j is a term in Stern's Diatomic Sequence. Note that this function is consistent with the rules defining Stern's Sequence since

$$f(2k/2^{n+1}) = \frac{a_{2k}}{a_{2^{n+1}+2k}} = \frac{a_k}{a_{2^n+k}} = f(k/2^n)$$

Theorem *The function f defined above extends to a strictly increasing continuous function from the unit interval to itself.*

Proof. Consider the matrices defined by terms with indices m, n in Stern's Diatomic Sequence:

$$M(m,n) := \begin{pmatrix} a_{m+1} & a_{n+1} \\ a_m & a_n \end{pmatrix}$$

and define:

$$A_0 := M(0,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note that $A_0M(m,n) =$

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_{m+1} & a_{n+1}\\ a_m & a_n\end{pmatrix} = \begin{pmatrix}a_{m+1}+a_m & a_{n+1}+a_n\\ a_m & a_n\end{pmatrix}$$

And similarly, $A_1M(m,n) =$

$$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}a_{m+1} & a_{n+1}\\ a_m & a_n\end{pmatrix} = \begin{pmatrix}a_m & a_n\\ a_{m+1}+a_m & a_{n+1}+a_n\end{pmatrix}$$

From the definition of the Diatomic Sequence, we know that $a_{2n+1} = a_{n+1} + a_n$. Therefore, for $i = 0$ or 1, $A_iM(m,n) = M(2m+i, 2n+i)$.

Hence, if we let $m = \sum_{k=0}^{n} i_k 2^k$, then by induction on n:

$$A_{i_0}A_{i_1}...A_{i_n}M(0,1) = M(m, 2^{n+1}+m)$$

Hence, for positive integers n and $k < 2^n$, $M(k, 2^n+k)$ is a product of matrices each with determinant 1 and for all n and $k < 2^n$ we have,

$$a_{k+1}a_{2^n+k} - a_k a_{2^n+k+1} = 1$$

With this, we note that each odd-indexed term in the nth row of Stern's Diatomic Array (see page 5) is the sum of two consecutive entries in the previous row, so it is at least one more than an odd entry of the previous row. Hence, for all n and $k < 2^n$,

$$a_{2^n+k}a_{2^n+k+1} \geq n+1$$

From the above two statements and [4] we have,

$$\frac{a_{k+1}}{a_{2^n+k+1}} - \frac{a_k}{a_{2^n+k}} = \frac{a_{k+1}a_{2^n+k} - a_k a_{2^n+k+1}}{a_{2^n+k}a_{2^n+k+1}} = \frac{1}{a_{2^n+k}a_{2^n+k+1}}$$

Hence,

$$0 < \frac{a_{k+1}}{a_{2^n+k+1}} - \frac{a_k}{a_{2^n+k}} \leq \frac{1}{n+1}$$

From this we see that for a given k, the difference between $f((k+1)/2^n)$ and $f(k/2^n)$ is bounded above by 1/(n+1) and always positive. Therefore, our function f extends to a strictly increasing function: $[0,1] \to [0,1]$.

$\square$

For more information about binary representations of n and more complex examples of matrix computations with a_n, the more curious reader should see Allouche and Shallit [3].

7.3 An Inverse Definition

Our function $f(x)$ is also known as Conway's box function which is notated as $\boxed{x}$ has a continuous inverse since we have shown it is strictly increasing. This inverse function, $f^{-1}(x)$ is Minkowski's question mark function.

Theorem $f^{-1}(x) = ?(x)$

Proof. By the definition of ?(x) (see page 11), ?(x) is uniquely determined by $?(1) = 1$ and

$$?(\frac{1}{n+x}) = \frac{2-?(x)}{2^n}$$

For our proof, it is sufficient to show that $f^{-1}(x)$ satisfies these equations, or equivalently, $f(1) = 1$ and

$$f(\frac{2-x}{2^n}) = \frac{1}{n+f(x)}$$

Since f maps D (the set of dyadic rationals) to $\mathbb{Q}^+$, our task is to show that the above is true for $x \in D$ (ie $x = k/2^m$).

By the palindromic nature of Stern's Diatomic Array (see page 5), we have

$$a_{2^{m+1}+2^{m+1}-k} = a_{2^{m+2}-k} = a_{2^{m+1}+k} = a_k + a_{2^{m+1}-k}$$

and

$$a_{2^{m+n+1}+2^{m+1}-k} = a_{2^{m+n}+2^{m+1}-k} + a_{2^{m+1}-k}$$

Therefore, by induction on n,

$$a_{2^{m+n}+2^{m+1}-k} = a_k + na_{2^{m+1}-k} = a_k + na_{2^m+k}$$

Then we can simply plug these values in for $f(k/2^n) = \frac{a_k}{a_{2^n+k}}$, and apply the equalities above:

$$f(\frac{2-k/2^m}{2^n}) = f(\frac{2^{m+1}-k}{2^{m+n}})$$

$$= \frac{a_{2^{m+1}-k}}{a_{2^{m+n}+2^{m+1}-k}} = \frac{a_{2^m+k}}{a_k + na_{2^m+k}} = \frac{1}{n+f(k/2^m)}$$

Hence, our function $f^{-1}(x)$ satisfies the same properties which uniquely determine $?(x)$. [1]

□

8 Conclusions

This ends our brief exploration of Stern's Diatomic Sequence, Array, and applications as presented by Northshield. The especially interested reader should consider the references and future directions of inquisition listed below.

There are many additional results and possible extensions that have either not been fully described here, or are even yet to be understood. One such conclusion is conjectured here for the more adventurous reader to ponder.

The "Fibonacci Diatomic Sequence" gives rise to the *Fibonacci Diatomic Array* and its associated *hyperbolic graph* $S_{3,2}$. What are some analogues between this and Stern's sequence and/or the hyperbolic tilings? Are there any further generalizations?

The Fibonacci sequence satisfies $F_n = \Theta(\phi^n)$, meaning F_n/ϕ^n is bounded away from 0 and ∞, as does the Diatomic sequence with $a_n = \Theta(\phi^{\log_2 n})$ despite some uncertainties (see [1, p. 588]). More generally, other diagonal sums across Pascal's triangle satisfy the following as presented by Northshield:

$$\sum_{ai+bj=n} \binom{i+j}{i} = \Theta(\gamma^n)$$

here γ is the unique positive solution to $\gamma^a + \gamma^b = \gamma^{a+b}$ (see another article by Northshield for more details [**4**]). For the same $a, b, and \gamma$, could the following be true?

$$\sum_{ai+bj=n} \binom{i+j}{i} mod 2 = \Theta(\gamma^{\log_2 n})$$

What if we considered Pascal's triangle mod 3 instead?

There are many sequences which are deeply related to Stern's Diatomic Sequence and have just as many fascinating properties. One such example the reader may find interesting is the *Thue-Morse* sequence which is discussed in great depth in Allouche and Shallit's work [**5**]. These sequences are remarkably similar in applications despite posessing different definitions. It is fascinating that even the simplest sequences defined within the discrete branches of mathematics can yield the most elegant results and rare complex richness.

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Problem. *Hello World!*

Proof. □