The Generic Hausdorff Dimension of Images of Continuous Functions

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1 Introduction

We begin with $C(X)$, the set of all real-valued continuous functions on an uncountable, compact metric space X. In their article titled *A note on the prevalent dimensions of continuous images of compact spaces*, J. M. Fraser and J. T. Hyde examine the Hausdorff and packing dimensions of the images of continuous functions defined on X [FH]. In particular, the article presents a result detailing about the "generic" behavior of images of continuous functions on X with respect to Hausdorff and packing dimension. The issue at hand is complex, as $C(X)$ is an infinite dimensional space and the usual theoretical means to establish genericity (such as Lebesgue measure) do not extent nicely to $C(X)$. Furthermore, continuous functions can often behave badly, further complicating possible approaches. A working theory of genericity is constructed and applied to the behavior of the dimensions, and the methods used in this application are surprisingly slick despite the complications mentioned above.

This paper intends to "unpack" both the setup and the proof of the main result of the article: The Hausdorff and packing dimensions of the images of "most" continuous functions on X are as large as they can be, namely 1. The first sections of the paper will provide the necessary mathematical background for understanding the statement of the main theorem as well as for understanding the definitions and theorems that we will use directly in the proof. The topics to be covered in these sections include basic measure theory (with a special emphasis on Borel sets), Hausdorff and packing dimension, and a rudimentary compilation of relevant topological notions. The span of topics required directly speaks to the sophistication of the issue at hand. The section on "Prevalence" constructs the theoretical means with which statements such as "for almost all" can be made in a large space such as $C(X)$. Finally, we give a formal statement of the theorem, as well as a detailed proof. Hopefully the reader can find interest in the theory leading up to the proof, as well as in the constructs and techniques used within the proof itself.

2 Definitions and Theoretical Background

2.1 Borel Sets and Measure Theory

A strong intuition of some basic properties of Borel sets is critical before commencing, as it is in the terminology of Borel sets that the notion of abstract "measurability" of a set is articulated. Further along the line, the Hausdorff and packing dimensions will be defined in terms of measures. Finally, more advanced measure-theoretic techniques will be used extensively in the proof, and we will refer back to this section whenever necessary.

A class $\mathbb{S}$ of subsets of a space S is called an *algebra* if the following conditions are satisfied:

1. $\varnothing \in \mathbb{S}$.
2. The class $\mathbb{S}$ is closed under complementation: if $A \in \mathbb{S}$, then $A^C \in \mathbb{S}$.
3. The class $\mathbb{S}$ is closed under finite unions: finite unions of sets in $\mathbb{S}$ are in $\mathbb{S}$.
4. The class $\mathbb{S}$ is closed under finite intersections: finite intersections of sets in $\mathbb{S}$ are in $\mathbb{S}$.

Furthermore, an algebra $\mathbb{S}$ of subsets of a space S is called a *σ-algebra* if $\mathbb{S}$ contains the limit of every monotone sequence of its sets. The pair $(S, \mathbb{S})$ is called a *measurable space*, and the sets in $\mathbb{S}$ are *measurable*. We let $\mathbb{F}$ and $\mathbb{G}$ denote the open and closed subsets of a topological space respectively. The class $\mathbb{B}(S)$ of *Borel subsets* of a metric space S is the σ-algebra $\sigma(\mathbb{G})$ $(= \sigma(\mathbb{F}))$. The equivalence $\sigma(\mathbb{G}) = \sigma(\mathbb{F})$ can be explained intuitively using the elementary properties of open and closed sets: every closed set in a metric space is the countable intersection of open sets, and complementation yields that every open set in a metric space is a countable union of closed sets. This is only an aside, however.

It may momentarily seem that our examination of Borel sets is delving into irrelevant and abstract rigor. Yet from our above formulation of Borel subsets in terms of σ-algebras, we can simply read off the following implications: the class of *Borel sets* is the smallest collection of subsets of $\mathbb{R}^n$ with the following properties:

1. every open set and every closed set is a Borel set;
2. the union of every finite or countable collection of Borel sets is a Borel set, and the intersection of every finite or countable collection of Borel sets is a Borel set.

This definition is certainly more approachable as far as getting an intuitive sense of what Borel sets are, as we see that any set constructed from a sequence of countable unions or intersections starting with the open sets or closed sets will be Borel. The class of Borel sets is indeed very large. Finally, we say that a measure μ is a *Borel measure* if μ is defined on the class of Borel subsets of a metric space.

A function μ is called a *measure* on a σ-algebra $\mathbb{S}$ if μ assigns a nonnegative number, possibly ∞, to each subset of $\mathbb{S}$ such that:

1. $\mu(\varnothing) = 0$;
2. $\mu(A) \leq \mu(B)$ if $A \subset B$ (*monotonicity*);
3. if $A_1, A_2, ...$ is a countable (or finite) sequence of sets then
$$\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$$
with equality if and only if the A_i are disjoint Borel sets.

Combining a few of the definitions developed so far, if μ is a measure defined on the σ-algebra $\mathbb{S}$ of subsets of S, the triple $(S, \mathbb{S}, \mu)$ is a *measure space*, and the sets in $\mathbb{S}$ are *μ-measurable.* The following property of measures will prove to be useful later on in the paper: if, for $\delta{>}0$, A_δ are Borel sets that are increasing as δ decreases ($A_{\delta'} \subset A_\delta$ for $0{<}\delta{<}\delta'$), then

$$\lim_{\delta \to 0} \mu(A_\delta) = \mu(\bigcup_{\delta > 0} A_\delta)$$

The *support* of a measure μ, written spt μ, is the smallest closed set X such that $\mu(\mathbb{S} \setminus X) = 0$. We say that μ is a measure *on* a set A if $A \supset \text{spt}\mu$. Informally, we can think of the support of a measure as the set on which the measure is concentrated. We refer to a measure μ on a bounded subset of $\mathbb{R}^n$ for which $0{<}\mu(\mathbb{R}^n){<}\infty$ as a *mass distribution.*

We provide a short formulation of the Lebesgue measure, without proving supplementary theorems. It is assumed that the readers are familiar with the notion of Jordan Measure, and should be able to spot the parallels between the two definitions. We begin with a "coordinate parallelopiped" in $\mathbb{R}^n$ described as

$$A = \{(x_1, ..., x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}.$$

The volume of the parallelopiped is given by

$$vol^n(A) = (b_1 - a_1)(b_2 - a_2)...(b_n - a_n).$$

We define the *Lebesgue measure* on $\mathbb{R}^n$ as

$$\mathcal{L}^n = \inf \left\{ \sum_{i=1}^{\infty} \text{vol}^n(A_i) : A \subset \bigcup_{i=1}^{\infty} A_i \right\}$$

where the infimum is taken over all covers of A by coordinate parallelopipeds A_i.

We also need a working definition of integration. The following definitions are not as rigorous as possible, but there is no need as we will only see a measure integral once, and we will not evaluate it directly as per the definition. A function $f : D \to \mathbb{R}$ is a *simple function* if it takes only finitely many values $a_1, ..., a_k$. The *integral with respect to the measure* μ of a simple function f is defined as

$$\int f d\mu = \sum_{i=1}^{k} a_i \mu \{x : f(x) = a_i\}$$

Now, if $f : D \to \mathbb{R}$ is a non-negative function, we define its integral as

$$\int f d\mu = sup \left\{ \int g d\mu : g \text{ is simple}, 0 \leq g \leq f \right\}$$

And finally, if f takes both positive and negative values, let $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \max\{-f(x), 0\}$. So $f = f_+ - f_-$, and we define

$$\int f d\mu = \int f_+ d\mu - \int f_- d\mu$$

The similarity to the buildup of the definition of the Riemann integral should be noted.

2.2 Hausdorff and Packing Dimensions

With our measure-theoretic background, we can now develop and motivate the main concepts at stake in this paper: those of Hausdorff and packing dimension. Our involvement will not extend far beyond the basic definitions, and for an expansive treatment of these notions the reader may consult [F].

For $U \in \mathbb{R}^n$, the *diameter* of U is defined as $|U| = \sup\{|x - y| : x, y \in U\}$. If $F \in \mathbb{R}^n$ and if $F \subset \bigcup_{i=1}^{\infty} U_i$ where $0 \leq |U_i| \leq \delta$, the collection of sets $\{U_i\}$ is called a δ*-cover* of F. We now consider all covers of of F by sets of diameter at most δ, and minimize the sum of the sth power of the diameters:

$$\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$$

This is often referred to as the "approximating Hausdorff measure." The *s-dimensional Hausdorff measure* $\mathcal{H}^s(F)$ is defined to be the limit as $\delta \to 0$ of $\mathcal{H}^s_\delta(F)$:

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$$

Naturally, coverings of F by small balls of radii no bigger than δ are permissible in our definition of the Hausdorff measure, and for our purposes it is intuitively easier to think of the covering $\{U_i\}$ as a covering of F by small balls. It can be shown that calculating the Hausdorff measure (and correspondingly Hausdorff dimension) using balls of radius at most δ is equivalent to using arbitrary sets of diameter at most δ, though this is beyond the scope of the paper.

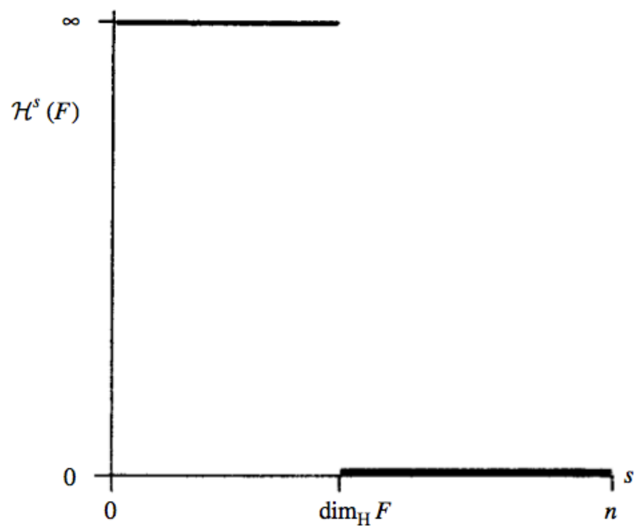

Figure 1: A graph of the Hausdorff measure against s for a set F

Henceforth the reader can think of the Hausdorff measure and dimension as involving coverings by small balls.

Examining equation (9), for an arbitrary $F \subset \mathbb{R}^n$ and $\delta<1$, $\mathcal{H}^s_\delta(F)$ is non-increasing with s. Thus $\mathcal{H}^s(F)$ as defined in equation (10) is also non-increasing. Suppose $t>s$ and $\{U_i\}$ is a δ-cover of F. Then by simple algebraic manipulations, we obtain:

$$\sum_i |U_i|^t \leq \sum_i |U_i|^{t-s}|U_i|^s \leq \delta^{s-t} \sum_i |U_i|^s$$

We take infima of both sides, obtaining that $\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F)$. Letting $\delta \to 0$, it is clear that if $\mathcal{H}^s(F)<\infty$ then $\mathcal{H}^t(F) = 0$ for $t>s$. Intuitively, there is a critical value of s at which $\mathcal{H}^s(F)$ "jumps" from ∞ to 0. This is illustrated in Figure (1). We call this critical value the *Hausdorff dimension* of F, denoted $\dim_H F$. More formally, we define the Hausdorff dimension as:

$$\dim_H F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$$

Thus $\mathcal{H}^s(F) = \infty$ if $0 \leq s < \dim_H F$ and $\mathcal{H}^s(F) = 0$ if $s > \dim_H F$. If $s = \dim_H F$, then $\mathcal{H}^s(F)$may be zero or infinite, or may satisfy $0<\mathcal{H}^s(F)<\infty$. If F is a Borel set satisfying this latter condition, F is said to be and *s-set*. An important property of the Hausdorff dimension we can immediately derive is *monotonicity*: if $E \subset F$ then $\dim_H E \leq dim_H F$. This follows directly from the monotonicity of measures. We provide the following simple example of a Hausdorff dimension calculation for clarification purposes:

Example 1. *Let F be a flat disk of unit radius in $\mathbb{R}^3$. Applying the definition of the Hausdorff measure, by inspection we can see that $\mathcal{H}^1(F) = length(F) = \infty$. Furthermore, $0<\mathcal{H}^2(F) = (4/\pi)area(F) = 4<\infty$. And finally, $\mathcal{H}^3(F) = (6/\pi)volume(F) = 0$. So $\mathcal{H}^s(F) = \infty$ if $s<2$ and $\mathcal{H}^s(F) = 0$ if $s>2$. Thus we obtain* $\dim_H F = 2$.

The packing measure provides a different approach to calculating measures. Whereas the Hausdorff measure utilizes coverings by small balls, the packing measure utilizes dense "packings" of a set by disjoint balls of differing small radii. Suppose $\{B_i\}$ is a collection of disjoint balls of radii at most δ with centers in F. Define

$$\mathcal{P}^s_\delta(F) = \sup \left\{ \sum_{i=1}^{\infty} |B_i|^s \right\};$$

$$\mathcal{P}^s_0(F) = \lim_{\delta \to 0} \mathcal{P}^s_\delta(F).$$

We define the *packing measure* as

$$\mathcal{P}^s(F) = \inf \left\{ \sum_i \mathcal{P}^s_0(F_i) : F \subset \bigcup_{i=1}^{\infty} F_i \right\}$$

where we have decomposed F into a countable number of pieces $F_1, F_2, ...$ For clarification, we have decomposed F as above because $\mathcal{P}^s_0(F)$ is not a measure. For instance, if we consider countable dense sets then condition 3 in our definition of a measure is violated. However, it is verifiable that $\mathcal{P}^s(F)$ is in fact a measure as per the given definition. We now define the *packing dimension* of a sef F, denoted $\dim_P F$, like we have defined the Hausdorff dimension:

$$\dim_P F = \sup\{s : \mathcal{P}^s(F) = \infty\} = \inf\{s : \mathcal{P}^s(F) = 0\}$$

The packing dimension is in many ways a "dual" to the Hausdorff dimension, and as such many results in mathematics are presented for Hausdorff and packing dimensions simultaneously. As such, we have developed the mechanics for the definition of both models, and the theorem we shall prove specifies both. In the proof of the theorem, and henceforth, we will work exclusively with the Hausdorff measure as it is simply easier to work with.

We now introduce a means of calculating the Hausdorff dimension that reduces the computation to the verification of the convergence of an integral. For $s \geq 0$ the *s-potential* at a point x of $\mathbb{R}^n$ due to the mass distribution μ on $\mathbb{R}^n$ is defined as

$$\phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s}$$

The *s-energy* of μ is defined as

$$I_s(\mu) = \int \phi_s(x) d\mu(x) = \iint \frac{d\mu(x) d\mu(y)}{|x-y|^s}$$

We state the following theorem connecting the Hausdorff dimension of a set F to the s-energy of measures with support contained in F:

Theorem 1. *Let $F \subset \mathbb{R}^n$. If there is a mass distribution μ on F with with $I_s(\mu){<}\infty$ then $\mathcal{H}^s(F) = \infty$ and $\dim_H F \geq s$. Conversely, if F is a Borel set with $\mathcal{H}^s(F){>}0$ then there exists a mass distribution μ on F with $I_t(\mu){<}\infty$ for all $0{<}t{<}s$.*

The proof of this theorem entails a relatively involved process of estimating measures of small balls and taking limits. We will simply defer the proof to theorem 4.13 in [F]. We will use this theorem later on to prove a crucial technical result relating the Hausdorff dimension of two useful image spaces: for a continuous injection ϕ on X and for an arbitrary continuous f on X, we will show that for almost all real constants c, the Hausdorff dimension of the set $(f + c\phi)(X)$ is at least as large as the Hausdorff dimsnsion of the set $\phi(X)$.

2.3 Basic Topological Background

We take an abrupt shift away from measures and dimension to develop some rudimentary familiarity with metric spaces and topological spaces. The topological background provided here serves a multiple purpose. It should build up a sufficient repertoire of definitions for understanding the hypotheses placed on the spaces X and $C(X)$ in our principle theorem, as well as an intuition for some of the proof techniques we will be using that will involve a heavy reliance on topological notions.

A *metric space* (X, d) is a set X equipped with a metric d on X. A *metric* on a set X is a real-valued function d on $X \times X$ that has the following properties:

1. $d(x, y) \geq 0, \qquad x, y \in X.$
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x), \qquad x, y \in X.$
4. $d(x, z) \leq d(x, y) + d(y, z), \qquad x, y, z \in X.$

A metric space X is *complete* if every Cauchy sequence in X converges. We define a *norm* on a vector space $\mathfrak{X}$ as a function $x \to ||x||$ from $\mathfrak{X}$ to $\mathbb{R}$ that satisfies the following conditions:

1. $||x|| \geq 0, \qquad$ with equality if and only if $x = 0$.
2. $||cx|| = |c| ||x||, \qquad x \in \mathfrak{X}$, c scalar.
3. $||x + y|| \leq ||x|| + ||y||, \qquad x, y \in \mathfrak{X}.$

The space $\mathfrak{X}$ is called a *normed linear space* or *normed space* for short. We define the *supremum norm* as follows:

$$||x||_\infty = max(|x_1|, ..., |x_n|)$$

The verification that $||x||_\infty$ is a norm is straightforward. What is particularly useful about the supremum norm is that it can unproblematically be applied to infinite-dimensional spaces. Let $\mathfrak{X}$ be a normed space. Then

$$d(x, y) = ||x - y||, \qquad x, y \in \mathfrak{X}$$

is a metric on $\mathfrak{X}$. The verification of this is trivial. Combining a few of the above definitions, we obtain the definition of a *Banach space* as a normed vector space that is complete in that norm's metric.

We now generalize some of the properties of metric spaces explained above by looking at the more abstract case of "topological spaces." The ideas presented about metric spaces generalize notions of distance, and the ideas that we will formulate about topological spaces take that generalization one step further, expressing the notion of "closeness" in terms of open sets. And indeed, considering this informal relation between metric spaces and topological spaces, we can intuitively guess that metric spaces are examples of topological spaces, and the theory we will develop here applies directly to metric spaces.

Let X be a set. A family $\mathfrak{T}$ of subsets of X is a *topology* for X if $\mathfrak{T}$ has the following properties:

1. Both X and the empty set belong to $\mathfrak{T}$.
2. Any union of sets in $\mathfrak{T}$ belongs to $\mathfrak{T}$.
3. Any finite intersection of sets in $\mathfrak{T}$ belongs to $\mathfrak{T}$.

A *topological space* is a pair $(X, \mathfrak{T})$ where X is a set and $\mathfrak{T}$ is a topology for X. The topology induced by the metric $d(x, y) = ||x - y||$ is called the *norm topology* of $\mathfrak{X}$, and a sequence $\{x_j\}_{j=1}^{\infty}$ is said to *converge in norm* to $x \in \mathfrak{X}$ if it converges in the norm topology, that is, if

$$\lim_{j \to \infty} ||x_j - x|| = 0.$$

The norm characterizes subsets of X with a definition of convergence (among other properties), and we can intuitively see how such a characterization satisfies the three properties of a topology for X. A *completely metrizable topological vector space* is a vector space, X, for which there exists a metric, d, on X such that (X, d) is complete and the vector space operations are continuous with respect to the topology induced by d. We define a topological space X to be *compact* if every open cover of X has a finite subcover. A space X is *normal* if for each pair E and F of disjoint closed subsets of X, there exist disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$. A space X is *separable* if it contains a subset U such that $\bar{U} = X$. We say U is *dense* in X.

As a last bit of technical terminology, we define a *homeomorphism* from at topological space X to a topological space Y to be a function $f : X \to Y$ that is one-to-one and onto, such that a subset U is open in X if and only if $f(U)$ is open in Y. This latter condition is equivalent to asserting that both f and f^{-1} are continuous. The composition of homeomorphisms is a homeomorphism, and the inverse of a homeomorphism is also a homeomorphism. The spaces X and Y are *homeomorphic* if there exists a homeomorphism of X and Y. The property of being homeomorphic is in fact an equivalence relation on the family of topological spaces, and in the topological context homeomorphic spaces are equivalent.

As stated in the introduction, the principal space that we will be concerned with and working within is $C(X)$, the space of real-valued continuous functions on an uncountable compact metric space X. We couple the supremum norm with $C(X)$, and observe that the resulting pair $(C(X), ||\,.\,||)$ is not only a Banach space but a completely metrizable topological vector space. We will soon see that this is the necessary prerequisite for being able to formally show the "genericity" of a set as we shall define it in the following section.

3 Prevalence

With a firm grasp of some the basic definitions of topology, we can now develop the principal machinery we will be using to prove the genericity of the set we are interested in: the set of functions on X with positive Hausdorff dimension whose images have Hausdorff and packing dimension equal to 1. The need for this new theory arises out of the ineffectiveness of Lebesgue integration for providing a means of formalizing statements such as "for almost all" for our specific purpose. In most contexts, it is usually sufficient to say that an assertion about points in a measurable space holds *almost everywhere* on the space if the set where the assertion is false is a set of Lebesgue measure zero. However, there is no analogue to Lebesgue measure in an infinite dimensional space such as $C(X)$. So instead of constructing a different measure (which is indeed possible), we construct a theory of genericity that works with finite dimensional subsets of a space. A sound theory of genericity should satisfy the *genericity axioms*:

Axiom I A generic subset of X is dense in X.

Axiom II If $L \supset G$ and G is generic, then L is generic.

Axiom III A countable intersection of generic sets is generic.

Axiom IV Every translate of a generic set is generic.

Axiom V A subset G of $\mathbb{R}^n$ is generic if and only if G is a set of full Lebesgue measure in $\mathbb{R}^n$.

For clarification, the *translate* of a set S is the set $S + x = \{s + x : s \in S\}$. A set has *full Lebesgue measure* in $\mathbb{R}^n$ if its complement has Lebesgue measure zero. We proceed to define prevalence:

Definition 1. *Let X be a completely metrizable topological vector space. A Borel set $E \subseteq X$ is* prevalent *is there exists a Borel measure μ on X such that:*

1. $0<\mu(C)<\infty$ *for some compact subset C of X,*

2. *The set $E+x$ has full μ-measure for all $x \in X$:*

$$\mu(X \setminus (E+x)) = 0$$

The complement of a prevalent set is called a *shy* set. We can think of the measure μ in this definition as the Lebesgue measure concentrated on a finite dimensional subset of X. We will not go through the proof that the theory of prevalence satisfies the axioms of genericity. It will suffice to remark that this is in fact the case, and for a detailed proof the readers may consult [OY]. The following definition gives us a direct means of computing the prevalence of a set:

Definition 2. *A finte-dimensional subspace $P \subseteq X$ is said to be a* probe *for a set $F \subseteq X$ if there exists a Borel set $E \subset F$ such that $E+x$ has full $\mathcal{L}_P$-measure for all $x \in X$. That is,*

$$\mathcal{L}_P(X \setminus (E+x)) = 0$$

where $\mathcal{L}_P$ denotes Lebesgue measure on the finite-dimensional subspace P. The existence of a probe is a sufficient condition for a set F to be prevalent. We say that $F \subset X$ is k-prevalent *if there exists a k-dimensional probe for F.*

Notice that $\mathcal{L}_P$ is in fact a restriction of the Lebesgue measure to P. We provide a very simple example of how the method of constructing a probe works to show the prevalence of a set.

Example 2. *Almost every function $f \in L^1[0,1]$ satisfies*

$$\int_0^1 f(x)dx \neq 0.$$

To show this, we construct P to be the one-dimensional subspace of $L^1[0,1]$ consisting of the constant functions. Now,

$$\int_0^1 (f(x)+\alpha)dx = 0$$

for one and only one value of α, namely $\alpha = -\int_0^1 f(x)dx$. Because

$$\mathcal{L}_P(\left\{\alpha \in \mathbb{R} : \int_0^1 (f(x)+\alpha)dx = 0\right\}) = 0$$

for every $f \in L^1[0,1]$, P is thus a probe, and the set of Lebesgue-integrable functions in $[0,1]$ that do not integrate to zero is a prevalent set in $[0,1]$.

We are now ready to provide a detailed account of the results of the article, as well as a complete proof.

4 Theorem

The machinery and theory that we have built up so far will be combined to yield the following result:

Let X be an uncountable compact metric space and let $C(X)$ denote the set of real-valued continuous functions on X. Then the set of functions $f \in C(X)$ for which $\dim_H f(X) = \dim_P f(X) = 1$ is a 1-prevalent set.

5 Proof of Theorem

The proof will start by showing that the set $\{f \in C(K) : \dim_H f(K) = 1\}$ is a Borel subset of $C(K)$. This is a crucial result, as it yields that our set satisfies the Borel hypotheses of Theorem 1 and Definition 2. We will then use Theorem 1 to derive a useful bound on the Hausdorff dimension of linear translates of a function. We will then show that $\{f \in C(K) : \dim_H f(K) = 1\}$ admits a probe and is thus prevalent in $C(X)$.

Lemma 1. *For a compact subset K of X, where X is a complete separable metric space, the set $\{f \in C(K) : \dim_H f(K) = 1\}$ is a Borel subset of $C(K)$.*

Proof: We will first prove that for a complete separable metric space X, the function $\dim_H : \mathfrak{K}(X) \to [0, \infty]$ is of *Baire class 2*. A *separable* metric space is simply one for which there is a dense subset that is countable. For reference, we use the standard definition that *Baire class* 0 consists of all continuous functions, and *Baire class* $n+1$ consists of all point-wise limits of sequences of functions in Baire class n. The proof below will start by showing that $\mathcal{H}^s_\delta : \mathfrak{K}(X) \to [0, \infty]$ is upper semicontinuous (in fact, we can also say that it is of Baire class 1, though this will not be proven). This will then yield that $\dim_H : \mathfrak{K} \to [0, \infty]$ is of Baire class 2. We will then cite a theorem proven in [D] that states that a function from a metric space into $[0, \infty]$ is a Baire function if and only if the function is Borel measurable, and thus conclude that $\dim_H$ is in fact Borel measurable. Once we have this, the fact that $\{f \in C(K) : \dim_H f(K) = 1\}$ is a Borel subset follows from an examination of the definitions of measurability and Borel measurability of a function.

Let $\mathfrak{K}(X)$ denote the set of all non-empty compact subsets of X. Define the distance between a point x and a subset Y of a metric space (M, d) by

$$\operatorname{dist}(x, Y) = \inf \{d(x, y) : y \in Y\}$$

We define the *Hausdorff metric* or *Hausdorff distance* between two subsets X and Y of a metric space (M, d) as

$$d_{\mathcal{H}}(X, Y) = \sup \{\operatorname{dist}(x, Y), \operatorname{dist}(y, X) : x \in X, y \in Y\}$$

It can be verified that $d_{\mathcal{H}}$ is a metric. Now, couple $\mathfrak{K}(X)$ with $d_{\mathcal{H}}$. The space $(\mathfrak{K}(X), d_{\mathcal{H}})$ is a metric space. Thus $\dim_H : \mathfrak{K}(X) \to [0, \infty]$ is a function that maps a metric space to $[0, \infty]$. We will now show that this function is in fact of Baire class 2.

We start by proving $\mathcal{H}^s_\delta$ is upper semicontinuous. The appropriate generalization of the metric space definition of continuity to a topological space is worded in terms of open sets. A function $f : X \to Y$ is *continuous at a point* $x \in X$ if for every open set V in Y such that $f(x) \in V$, there exists an open set U in X such that $x \in U$ and $f(U) \subseteq V$. Now clearly a function is *continuous* on a domain X if and only if it is continuous at every point of X. We say that a function f is *upper semicontinuous at a point* x_0 if for every $\epsilon{>}0$ there is a neighborhood U of x_0 such that $f(x) \leq f(x_0) + \epsilon$ for all $x \in U$. Finally, we say that a function f is *upper semicontinuous* if it is upper semicontinuous at every point of it's domain X, which is to say if and only if the set $\{x \in X : f(x){<}c\}$ is an open set for every $c \in \mathbb{R}$. The goal is thus to show that the set $\{K \in \mathfrak{K} : \mathcal{H}^s_\delta{<}c\}$ is open for all constants c.

Let $\mathfrak{B}$ be a *base for the topology* of X, meaning that every open subset of X is a union of sets in $\mathfrak{B}$. Let $\{W_n\}_{n=1}^{\infty}$ be an enumeration of all finite unions of the sets of $\mathfrak{B}$. Let c be a real constant and let $K \in \mathfrak{K}(X)$. By the compactness of K and the definition of the approximating Hausdorff measure, $\mathcal{H}^s_\delta(K){<}c$ if and only if there are finitely many open sets $U_1, ..., U_k$

$$\sum_{i=1}^{k} |U_i|^s{<}c, \quad K \subset \bigcup_{i=1}^{k} U_i \ \text{ and } \ |U_i|{<}\delta \ \text{ for } \ i = 1, ..., k.$$

Recall that $|U|$ denotes the diameter of the set U. Now, the condition specified in (26) holds if and only if there are indices $n_1, ..., n_m \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} |W_{n_i}|^s{<}c, \quad K \subset \bigcup_{i=1}^{m} W_{n_i} \ \text{ and } \ |W_{n_i}|{<}\delta \ \text{ for } \ i = 1, ..., m.$$

Referring back to the definition of a topology provided in an earlier section, if $\mathfrak{T}$ is a topology for a set X, then the sets in $\mathfrak{T}$ are called *open sets*. This is to say that a topology for X is a specification of certain subsets of X as "open" sets with the properties 1, 2, and 3 required. We would like to show that the set of $K \in \mathfrak{K}(X)$ for which there exist $n_1, ..., n_m$ satisfying (30) is open in some topology of $\mathfrak{K}(X)$. For this, examine the *Vietoris topology* generated as follows: for every n-tuple $U_1, ..., U_n$ of open sets in $\mathfrak{K}(X)$, we construct a basis set consisting of all subsets of $\bigcup_{i=1}^n U_i$ that have non-empty intersections with each U_i. It is left to the intuition of the reader to notice that $\{K \in \mathfrak{K}(X) : (30) \text{ is satisfied}\}$ is open in the Vietoris topology. We conclude that $\{K \in \mathfrak{K}(X) : \mathcal{H}^s_\delta(K){<}c\}$ is open, proving that $\mathcal{H}^s_\delta$ is upper semicontinuous.

Now we proceed to show that $\dim_H$ is a Baire class 2 function. Examining the set of F for which $\dim_H F \leq c$ and writing δ as $1/n$ in the approximating Hausdorff measure. We have the following:

$$\{F \in \mathfrak{K}(X) : \dim_H F \leq \alpha\} = \bigcap_{s>\alpha, s\in\mathbb{Q}^+} \bigcap_{n} \left\{F : \mathcal{H}^s_{1/n}(F){<}1\right\}$$

The above set is a countable intersection of open sets. It follows then that the set $\{F : \beta < \dim_H F \leq \alpha\}$ is a countable union of a countable intersection of open sets for all $0 > \beta > \alpha$. So it is not too much of a stretch to deduce that $\dim_H$ is the point-wise limit of a sequence of Baire 1 functions, and thus a Baire 2 function. For a more complete proof of this, the reader should consult [MM]. We cite the following theorem from [D]:

Theorem 2. *A function from a metric space into $\mathbb{R}$ is a Baire function if and only if the function is Borel measureable.*

$\dim_H$ is a Baire function that maps $\mathfrak{K}(X)$, a metric space when equipped with the Hausdorff metric, into $\mathbb{R}$. We conclude that $\dim_H$ is Borel measurable.

The rest of the proof follows quickly. We switch notation for convenience: Let $\Delta_H : (\mathfrak{K}(\mathbb{R}), d_{\mathcal{H}}) \to \mathbb{R}$ be defined by $\Delta_H(K) = dim_H K$. Notice that $\mathbb{R}$ is clearly complete and separable. For the latter property, we consider $\mathbb{Q}$, which is a countable dense subset of $\mathbb{R}$. Thus our above proof for an arbitrary complete separable metric space is still applicable. We define another function $\Lambda : C(K) \to \mathbb{R}$ by $\Lambda(f) = f(X)$, which is clearly continuous. Now, notice that we can express

$$\{f \in C(K) : \dim_H f(K) = 1\} = (\Delta_H \circ \Lambda)^{-1}(\{1\})$$

We turn to the definition of a measurable function: If $(S, \mathbb{S})$ is a measurable space, a function f from a set A in $\mathbb{S}$ into a measurable space $(S', \mathbb{S}')$ is *measurable* if $f^{-1}(\mathbb{S}') \subset \mathbb{S}$. Notice that $\mathbb{S}$ and $\mathbb{S}'$ are σ-algebras as per our definition. Furthermore, the definition of a Borel measurable function can be refined to the following: A measurable function from one metric space to another is *Borel measurable*. In particular, it can be easily derived that that a continuous function from one metric space into a second is Borel measurable, as the inverse image of an open set under a continuous function is open and the open sets generate the σ-algebra of Borel sets. Clearly the composition $\Delta_H \circ \Lambda$ is Borel measurable, and as such its image is the σ-algebra that generates the class of Borel sets. Thus $(\Delta_H \circ \Lambda)^{-1}(\{1\})$ is necessarily a Borel subset. $\square$

We now prove a lemma that gives us a very useful estimate on the Hausdorff dimension of the images of linear translates of the form $f + c\phi$ with respect to the Hausdorff dimension of the image of ϕ.

Lemma 2. *Let $\phi \in C(X)$ be an injective function, and choose an arbitrary $f \in C(X)$. Then for almost all $c \in \mathbb{R}$ we have*

$$\dim_H(f + c\phi)(X) \geq \dim_H \phi(X)$$

Proof: Prior to beginning, we note that the phrase "for almost all" used in the statement of the theorem is used in the Lebesgue measure context. Suppose $\phi \in C(X)$ is an injection. Fix $f \in C(X)$, fix $\epsilon \in (0, s)$. Let $s = \dim_H \phi(X) \leq 1$. By Proposition 3.1 there exists a Borel probability measure μ supported on $\phi(X)$ with finite $(s - \epsilon)$-energy:

$$I_{s-\epsilon}(\mu) < \infty$$

Since ϕ is a bijection between X and $\phi(X)$ we can define a measure $\nu = \mu \circ \phi$ on X. Now, for $c \in \mathbb{R}$ define $\nu_c = \nu \circ (f + c\phi)^{-1}$ be the image measure of ν under $f + c\phi$. Since ν_c is supported on $(f + c\phi)(X)$, it suffices to show that for almost all $c \in \mathbb{R}$ the measure ν_c has finite $(s - \epsilon)$-energy, i.e.

$$I_{s-\epsilon}(\nu_c)<\infty$$

For $c \in \mathbb{R}$ we have the following expression for $I_{s-\epsilon}(\nu_c)$.

$$I_{s-\epsilon}(\nu_{c,f}) = \int_{x\in f(K)} \int_{y\in f(K)} \frac{d\nu_c(x)d\nu_c(y)}{|x-y|^{s-\epsilon}}$$

We express $x = (f + c\phi) \circ (f + c\phi)^{-1}(x)$, $y = (f + c\phi) \circ (f + c\phi)^{-1}(y)$, and write $\nu_c = \nu \circ (f + c\phi)^{-1}$ in the integral:

$$= \int_{x\in f(K)} \int_{y\in f(K)} \frac{d(\nu \circ (f + c\phi)^{-1})(x)d(\nu \circ (f + c\phi)^{-1})(y)}{|(f + c\phi) \circ (f + c\phi)^{-1}(x) - (f + c\phi) \circ (f + c\phi)^{-1}(y)|^{s-\epsilon}}$$

Now we substitute $u = (f + c\phi)^{-1}(x)$, $v = (f + c\phi)^{-1}(y)$:

$$= \int_{u\in K} \int_{v\in K} \frac{d\nu(u)d\nu(v)}{|(f + c\phi)(u) - (f + c\phi)(v)|^{s-\epsilon}}$$

Expanding the denominator of the integrand, $|(f + c\phi)(u) - (f + c\phi)(v)|^{s-\epsilon} = |f(u) + c\phi(u) - f(v) - c\phi(v)|^{s-\epsilon} = |(f(u) - f(v)) + c(\phi(u) - \phi(v))|^{s-\epsilon}$. The integral has now been simplified to:

$$= \int_{u\in K} \int_{v\in K} \frac{d\nu(u)d\nu(v)}{|(f(u) - f(v)) + c(\phi(u) - \phi(v))|^{s-\epsilon}}$$

We wish to show that $I_{s-\epsilon}(\nu_c)$ is convergent. It is sufficient to simply integrate $I_{s-\epsilon}(\nu_c)$ from $-n$ to n and let $n \to \infty$, then show that this is finite:

$$\int_{-n}^{n} I_{s-\epsilon}(\nu_c) = \int_{-n}^{n} \int_{u\in K} \int_{v\in K} \frac{d\nu(u)d\nu(v)}{|(f(u) - f(v)) + c(\phi(u) - \phi(v))|^{s-\epsilon}} dc$$

Applying Fubini's theorem, which is permitted since the measures involved are finite, we switch the order of integration:

$$= \int_{u\in K} \int_{v\in K} \int_{-n}^{n} \frac{dc}{|(f(u) - f(v)) + c(\phi(u) - \phi(v))|^{s-\epsilon}} d\nu(u)d\nu(v)$$

We apply the following immediate bound:

$$\leq \int_{u\in K} \int_{v\in K} 2\int_{0}^{2n} \frac{dc}{c^{s-\epsilon}|\phi(u) - \phi(v)|^{s-\epsilon}} d\nu(u)d\nu(v)$$

Integrating through:

$$= 2\frac{(2n)^{1-s+\epsilon}}{1 - s + \epsilon} \int_{u\in K} \int_{v\in K} \frac{d\nu(u)d\nu(v)}{|\phi(u) - \phi(v)|^{s-\epsilon}}$$

Figure 2: The classical Cantor ternary set

$$\leq \frac{8n^2}{1-s+\epsilon}\int_{u\in K}\int_{v\in K}\frac{d(\mu\circ\phi)(u)d(\mu\circ\phi)(v)}{|\phi(u)-\phi(v)|^{s-\epsilon}}$$
$$= \frac{8n^2}{1-s+\epsilon}\int_{p\in\phi(K)}\int_{q\in\phi(K)}\frac{d\mu(p)d\mu(q)}{|p-q|^{s-\epsilon}}$$

Now we simply notice that the resulting expression on the right of the is the $s-\epsilon$ energy of the probability measure μ, which by Theorem 1 is finite since $0<s-\epsilon<s$. So to finish, the expression is equal to

$$\frac{8n^2}{1-s+\epsilon}I_{s-\epsilon}(\mu)<\infty$$

We have shown that $\dim_H(f+c\phi)(K) \geq s-\epsilon$ for almost all $c \in (-n,n)$. As a technical explanation, since we have integrated with respect to c there is a possible set of $c \in \mathbb{R}$ such that $I_{s-\epsilon}(\nu_c)$ is not finite. However, what he have shown is that the integral of this set is zero, and as such we can formally use the statement "for almost all." Hence $\dim_H(f+c\phi)(K) \geq \dim_H \phi(K)$ for almost all $c \in (-n,n)$. $\square$

We are now ready to complete the proof. Recall that our metric space X is compact, and as stated before we have that X is complete. Now, we cite Exercise 11: 3.6 in [BBT], which states that every complete metric space that is dense-in-itself contains a Cantor set, which we will denote K. A set is *dense-in-itself* if it contains no isolated points, and it should be evident that our compact space X is dense-in-itself. By *Cantor set* we mean any set that is homeomorphic to the classical Cantor ternary set. As a reminder, the *Cantor ternary set* is constructed by removing the middle thirds of a line segment. In particular, it can be shown that the Cantor ternary set has Lebesgue measure zero. The set is illustrated in Figure 2.

Taking compositions of homeomorphisms, it immediately follows that all Cantor sets as we have defined them are homeomorphic. We let $F \subset [0,1]$ be a Cantor set with Hausdorff dimension equal to 1, and appropriately name it a *fat Cantor set*. This may seem initially counter-intuitive, as we have previously mentioned that the real line $\mathbb{R}$ has Hausdorff dimension equal to 1. Nonetheless, such a set can be constructed. In the construction of the classical Cantor ternary set, the proportion 1/3 is removed from each remaining interval after each subsequent iteration in the construction. When constructing the fat Cantor set however, we would like the proportion removed from the remaining intervals after each iteration in the construction to be non-constant

Figure 3: The fat Cantor set constructed above

and decreasing, which is to say each iteration removes proportionally less from the remaining intervals. We start by removing the middle $1/4$ from the unit interval, and subsequently removing subintervals of width $1/2^{2n}$ from the middle of each of the 2^{n-1} remaining intervals. Thus intervals of total length $\sum_{n=0}^{\infty} 2^n(1/2^{2n+2}) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... = \frac{1}{2}$ are removed, and thus the remaining set has Lebesgue measure $\frac{1}{2}>0$. We will not derive that the fat Cantor set has Hausdorff dimension 1, though hopefully the above calculation of it's measure will make it seem at least plausible. For further elaboration, the reader may consult [AB]. For an illustration of a fat Cantor set, see Figure 3.

Moving on, since all Cantor sets are homeomorphic there exists a continuous bijection ϕ that maps K to F, where $F \subset [0,1] \subset \mathbb{R}$. We have slipped this under the rug a bit, as it is not immediate that there is a homeomorphism from the Cantor tertiary set to the fat Cantor set. To make this assertion rigorous, we cite Lemma 18.9 in [AB] as the following theorem:

Theorem 3. *For every* $0<\epsilon<1$ *there exists a continuous function* $f : [0,1] \to [0,1]$ *such that:*

1. f *is onto,*
2. f *is strictly increasing (and hence one-to-one),*
3. f *maps the* ϵ*-Cantor set* C_ϵ *onto the Cantor set* C*.*

ϵ*-Cantor set* is simply equivalent terminology for fat Cantor set. Note that the above function f is a homeomorphism; we have thus proven that our Cantor set K is homeomorphic to the fat Cantor set, and this homeomorphism is given by ϕ.

We would like to have a way of extending ϕ to all of X. This follows immediately from the well known Tietze Extension Theorem:

Theorem 4. *Let* E *be a normal topological space, let* F *be a closed subset of* E*, and let* f *be a bounded continuous real-valued function on* F*. Then there exists a bounded continuous real-valued function* h *on* E *such that* $h = f$ *on* F*.*

All metric spaces are normal, and thus X is normal. Furthermore the map ϕ is continuous by definition. The hypotheses of the theorem are satisfied, and we can thus extend ϕ to a continuous function which we will denote Φ on X. In particular, $\Phi \in C(X)$.

The last step of the proof involves constructing a probe for the set of continuous functions whose images have Hausdorff dimension 1. We claim that

$$P = \{c\Phi : c \in \mathbb{R}\}$$

is a probe for the set $A = \{f \in C(X) : \dim_H f(X) = 1\}$. Define $\pi_P : P \to \mathbb{R}$ by $\pi_P(c\phi) = c$. We can define the 1-dimensional Lebesgue measure on P through the composition $\mathcal{L}_P = \mathcal{L}^1 \circ \pi_P$. We fix an arbitrary $f \in C(X)$, and go through the computation verifying that P is indeed a probe as outlined in Definition 2 as well as in Example 2. We compute

$$\mathcal{L}_P(C(X) \setminus (A + f))$$

Recall that $\mathcal{L}_P$ is a restriction of the 1-dimensional Lebesgue measure to P. As such, we have the following:

$$\mathcal{L}_P(C(X) \setminus (A + f)) = \mathcal{L}_P(P \setminus (A + f))$$

Now, examining (46) the set $P \setminus (A + f)$ can be expressed equivalently as the set $\{c\Phi : c\Phi - f \notin A\}$. Writing $\mathcal{L}_P$ as $\mathcal{L}^1 \circ \pi_P$ as per definition, we have that

$$\mathcal{L}_P(P \setminus (A + f)) = (\mathcal{L}^1 \circ \pi_P)(\{c\Phi : c\Phi - f \notin A\})$$

Continuing the manipulations, we recognize that the negation of the condition $\dim_H f(X) = 1$ of the set A yields that

$$\{c\Phi : c\Phi - f \notin A\} = \{c\Phi \in C(X) : \dim_H(c\Phi - f)(X)<1\}\,.$$

The condition that $\dim_H <1$ is there because we have negated the statement $\dim_H = 1$ and it is clearly true that $\dim_H$ can be no more than 1 in our context. Furthermore, we apply the map π_P through the last set, and arrive at

$$(\mathcal{L}^1 \circ \pi_P)(\{c\Phi : c\Phi - f \notin A\}) = \mathcal{L}^1(\{c : \dim_H(c\Phi - f)(X)<1\})$$

The following "substitution" of sorts is a neat trick that arises out of our previous deductions regarding Cantor sets. Recall that we have shown that X contains a Cantor set K, and there is a homeomorphism ϕ that maps K to F where F is our fat Cantor set of Hausdorff dimension 1. This is to say that $\phi(K) = F$, and $\dim_H \phi(K) = \dim_H F = 1$. So we simply substitute $\dim_H \phi(K)$ for "1." Furthermore since $K \subset X$ and the Hausdorff dimension satisfies the property of monotonicity, $\dim_H(c\Phi - f)(X) \leq \dim_H(c\phi - f)(K)$, and as such the set $\{c : \dim_H(c\Phi - f)(X)<1\}$ is in fact contained in $\{c : \dim_H(c\phi - f)(K)<\phi(K)\}$. Now by the monotonicity property of the Lebesgue measure, we have the following:

$$\mathcal{L}^1(\{c : \dim_H(c\Phi - f)(X)<1\}) \leq \mathcal{L}^1(\{c : \dim_H(c\phi - f)(K)<\phi(K)\})$$

The proof is almost finished. By Lemma 2 we know that for an arbitrary $f \in C(X)$, for almost all $c \in \mathbb{R}$ we have $\dim_H(f + c\phi)(X) \geq \dim_H \phi(X)$, where we have noted that "for almost all" means "for all but a set of Lebesgue measure 0." Correspondingly, it immediately follows that

$$\mathcal{L}^1(\{c : \dim_H(c\phi - f)(K) < \phi(K)\}) = 0$$

So we have shown that $\mathcal{L}_P(C(X) \setminus (A + f)) = 0$. So by Lemma 1 and the fact that A admits a 1-dimensional probe as proven above, the set of functions $f \in C(X)$ for which $\dim_H f(X) = \dim_P f(X) = 1$ is a 1-prevalent set. □

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