Semantics With Algebra and Geometry

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1 Introduction

Historically, the standard semantics of modal logics were given by Saul Kripke in the 1960s in the form that is now known as 'Kripke Semantics.' These semantics, while very successful for investigating *propositional* modal systems, fail to accomodate first-order semantics in a 'natural' fashion. In particular, consistent extensions of many first-order modal systems (including many systems extending S4) fail to be complete on any Kripke-style models ([2] chapters 14 and 15.)

The main intention of this paper, then, is to give a substantial overhaul of semantics for S4 and its consistent extensions through the definition and application of sheaves and sheaf semantics. This framework is general enough and powerful enough to allow us to construct sound and complete models of predicate modal logics extending S4.

Outline of the Paper The paper is divided into two major sections: the first section deals with the topological semantics for S4 and the second section investigates the use of sheaves to define a more useful semantics for first order modal logic than given by Kripke semantics.

2 Topological Semantics for Propositional S4

In this section, the focus will be on bootstrapping a nonstandard semantics onto the modal system S4 by noting, as Awodey and Kishida [1] do, and exploiting a formal resemblance between modal operators and the interior and closure operators acting on topological spaces. To make the correspondence precise, I will Boolean homomorphisms in order to formally make the transformations go through.

2.1 The Syntactic System of Propositional S4

To begin with, we define the logical structure that we are investigating-propositional S4. First we introduce some basic conventions of mathematical logic, introducing the system of Propositional Calculus in order to define the modal system S4.

Definition A (modal) language $\mathcal{L}$ is a set of formulas consisting of propositional variables p_i for $i \in I$, logical constants, a constant $\top$ (to be thought of as "true") and well-formed formulas defined recursively in the form $\neg\alpha$, $\alpha \vee \beta$, and $\Box\alpha$ where α and β are variables, constants, or other recursively defined wffs.

What differentiates modal systems from non-modal systems is the inclusion of formulas like $\Box\alpha$. Intuitively, the box is to be read as a natural-language prefix such as "it is necessary that α" or "it is provable that α" or "it is believed that α." Now that we have a (modal) language, we can define the Propositional Calculus.

Definition The Propositional Calculus consists of the language $\mathcal{L}$ together with a binary derivability relation $\vdash$ satisfying the following rules for all $\alpha, \beta \in \mathcal{L}$: $\alpha \wedge (\neg\alpha \vee \beta) \vdash \beta$ (Modus Ponens), $\neg\neg\alpha \vdash \alpha$ (double negation), $\alpha_1 \wedge \cdots \wedge \alpha_n \vdash \alpha_i$ (and-elimination), $\alpha_i \vdash \alpha_1 \vee \cdots \vee \alpha_n$ (or-introduction), $\alpha_1, \cdots, \alpha_n \vdash \alpha_1 \wedge \cdots \wedge \alpha_n$ (and-introduction), $\alpha \vee \beta, \neg\beta \vdash \alpha$ (unit resolution), $\alpha \vee \beta, \neg\beta \vee \gamma \vdash \alpha \vee \gamma$ (resolution), $\alpha \vdash \top$ (material implication) and finally $\neg\beta \wedge (\neg\alpha \vee \beta) \vdash \neg\alpha$ (Modus Tollens).

With Propositional Calculus in mind, we define an extension of it called S4.

Definition The system S4 consists of a modal language $\mathcal{L}$ that satisfies the derivation rules of Propositional Calculus in addition to the following modal axioms: $\Box\alpha \vdash \alpha$, $\Box\alpha \vdash \Box\Box\alpha$, $\Box\alpha \wedge \Box\beta \vdash \Box(\alpha \wedge \beta)$, $\top \vdash \Box\top$, and $\alpha \vdash \beta$ implies $\Box\alpha \vdash \Box\beta$ where α and β are well-formed formulas of $\mathcal{L}$.

2.2 Elementary Topology

The main topological concepts required to make sense of the results are that of a topological space and the interior operator. These basic concepts will allow us to do two things: the interior operator will help to provide an adequate meaning for the logical concepts of 'model' and 'interpretation' for S4 and the almost bare-bones structure of a topological space is just structured enough to give us exactly what we need- a bounded lattice. For the time being, though, the only matters concerning us are the definitions and some quick corollaries about the interior operator.

Definition A topological space $X = (|X|, \mathcal{O}(X))$ is a pair consisting of a set $|X|$ together with a set of open sets $\mathcal{O}(X)$ satisfying the properties that the empty set and X are open, the finite intersection of two open sets is open, and the arbitrary union of open sets is open.

With this abstract definition, we go on to define the interior operator.

Definition We define the interior of $Y \in X$ to be $\text{int}(\cdot) = \underline{\cdot} : \mathcal{P}(X) \to \mathcal{P}(X)$ that sends $Y \mapsto \bigcup_{O \in \mathcal{O}(X) \subseteq Y} O$

Now, from this definition of the interior operator, we gain a few very easy corollaries that turn out to provide at least aesthetic justification for choosing the semantics as we did.

Corollary 2.1. *If X is a topological space then the interior operator $\underline{\cdot}$ satisfies*

1. $\underline{X} = X$
2. $\underline{Y} = \underline{\underline{Y}}$
3. $\underline{Y} \cap \underline{Z} \subseteq \underline{Y \cap Z}$
4. $Y \subseteq Z$ *implies that* $\underline{Y} \subseteq \underline{Z}$

Proof. 1. This falls right out of the definition of the interior.

2. We have by definition that $\underline{\underline{Y}} = \bigcup_{O \in \mathcal{O}(X) \subseteq \underline{Y}} O$. But if a set O_i is in the union and is distinct from $\underline{Y}$ then by assumption $O_i \subset \underline{Y}$ and so $\underline{Y}$ contains all points in the union, so the result follows.

3. Suppose that $w \in \underline{Y} \cap \underline{Z}$. Then for some $O_{i,Y}$ and $O_{j,Z}$ we have that $w \in O_{i,Y} \cap O_{j,Z}$. But $O_{i,Y} \cap O_{j,Z}$ is an open subset of $Y \cap Z$ so $w \in \underline{Y \cap Z}$.

4. Suppose that $Y \subseteq Z$. Then for any open O such that $O \subseteq Y$ $O \subseteq Z$. Hence, the union of all these open subsets is in Z so $\underline{Y} \subseteq \underline{Z}$.

□

Right off the bat, we have a group of corollaries that closely resemble the rules of derivation for our S4 system. This resemblance, which may at first seem to be very sketchy justification for some sort of transformation can in fact be formalized in the language of homomorphisms.

2.3 Welding Logic to Topology: Bounded Lattices, Homomorphisms, and Stone Spaces

In this section, we consider the underlying algebraic structure that unites the structure of topological spaces, the interior operation, and set inclusion on the one hand and languages, modality, and provability on the other. To do so, we will be working with algebraice structures called Boolean algebras.

Definition A Boolean Algebra is set P equipped with a relation $\leq$ such that for all $p, q, r \in P$

1. $p \leq p$

2. if $p \leq q$ and $q \leq p$ then $p = q$

3. if $p \leq q$ and $q \leq r$ then $p \leq r$

4. there are binary relations $\wedge$ and $\vee$ such that for all $p, q \in P$, $p \wedge q \leq p$ and $p \leq p \vee q$.

5. there exist elements $0, 1 \in P$ such that for all $p \in P$, $0 \leq p$ and $p \leq 1$.

6. there exists a unary operation $(\cdot)^c$ such that $p \vee p^c = 1$ and $p \wedge p^c = 0$

7. $\vee$ and $\wedge$ satisfy distributive laws.

Both S4 endowed with the derivability relation and topological spaces possess Boolean structure, as seen in the following lemma.

Lemma 2.2. *S4 is a Boolean algebra.*

Proof. S4 is already endowed with a binary relation $\vdash$, the symbol $\top$ and a new symbol (to be thought of as "false") $\bot =_{def} \neg\top$. Now, we can formally identify $\mathcal{L}$ with P, $\vdash$ with $\leq$, $\top$ with 1 and $\bot$ with 0. That this assignment gives us the algebra we want is easily checked; we have that $\alpha \vdash \alpha$ from the Propositional Calculus, the transitivity of $\vdash$ follows from the composability of proofs, we can define equality between two wffs by their interprovability, the "and" and "or" operations in S4 correspond to the wedge and vee of the algebra (and satisfy distributive laws), giving us local infimums and supremums, and that for all wffs $\alpha \vdash \top$ by the definition of material implication and $\bot \vdash \alpha$ by contraposition of $\neg\alpha \vdash \top$. Finally, negation acts as complementation and so all of the axioms are accounted for. $\square$

In a similar vein, the Boolean properties of topological spaces with respect to the subset relation can be proven in a straightforward manner.

Now, Boolean algebras obviously have some sort of "shared algebraic structure" that we would like to preserve in mappings between them. To make this notion precise, we define a class of maps called homomorphisms that leaves our algebraic structure invariant.

Definition A Boolean homomorphism $\hat{f} : (P, \leq) \to (Q, \prec)$ is a function $f : P \to Q$ such that if $p, q \in P$ satisfy $p \leq q$ then $f(p) \prec f(q)$ and also satisfies that the $0, 1, \vee, \wedge,$ and $(\cdot)^c$, of $(P, \leq)$ get mapped to their respective equivalents in $(Q, \prec)$.

With this definition in mind, we can see that we already have two operators that are homomorphisms- $\Box : \mathcal{L} \to \mathcal{L}$ sends $\alpha \mapsto \Box\alpha$ and the map $\text{int} : \mathcal{P}(X) \to \mathcal{P}(X)$ which takes $S \subseteq X \mapsto \underline{S} \subseteq X$.

To recapitulate, we have very closely related structures on topological spaces and on S4 structures, and we have two very closely related homomorphisms, $\Box$ and int. What we have yet to accomplish is a way to get the two to 'stick' together in some substantial way. The key to this, and to the proofs of soundness and completeness of S4 on some topological model, is a homomorphism $\|\cdot\|$ that takes elements of $\mathcal{L}$ to elements of $\mathcal{P}(X)$. The map that we will use for this purpose is called an *interpretation*. In classical model theory, interpretations take logical variables to truth values in the set $\{0, 1\}$ and take wffs to their corresponding evaluations (that is, we build up $\|\alpha\|$ by evaluating $\|x_i\|$ for each variable $x_i \in \alpha$ and then stringing them together by the wedge, vee, and complementation operations). In Topological Semantics, however, we instead map wffs of a language to certain subsets S of a topological space. This shift in and of itself is not so great in treating propositional semantics, however when we begin discussing first-order semantics we need to consider the notion of a sheaf, an inherently topological notion not adequately captured by the classical $0, 1$ semantics.

2.4 Topological Semantics of S4

In this subsection we consider two things. First, we investigate the prospect of generating a topological space that makes S4 complete with respect to it. The main result of this section will be the Stone Representation Theorem, which will put us in the position to prove the completeness of S4 with respect to the lattice is compact open subsets of S4's "Stone Space." The machinery required for these results is not unintuitive; they constitute certain generalizations of the standard methods of model theory. The construction we will use will have an interesting feature; in proving completeness, soundness falls out almost trivially. The main drawback with this approach is that in its generality, it is perhaps too weak- even though we will be able to find specific models and interpretation homomorphisms that make a system complete,

To begin with, let's reconsider the Boolean algebra that we associated with S4. We can do one further thing to simplify it: identify all provably equivalent statements with each other; that is if $\alpha \vdash \beta$ and $\beta \vdash \alpha$, then take $\alpha \sim \beta$. This is a quotient space of the Boolean algebra called the Lindenbaum algebra. The $\sim$ relation is, as [4] notes, an equivalence relation and a homomorphism of Boolean algebras. Let $\mathcal{S}4$ denote S4's Lindenbaum algebra, which identifies all α and β such that $\alpha \vdash \beta$ and $\beta \vdash \alpha$.

Now, let us define filters and ultrafilters on Boolean Algebras. The intuition behind ultrafilters is that they are very closely related to the maximal consistent sets used in model theory to define the so-called canonical models. The topology we produce as a complete model will, in fact, have as open sets ultrafilters of $\mathcal{S}4$. Now, a filter is characterized by the following property:

Definition A filter of a Boolean algebra $(P, \leq)$ is a subset $F \subseteq P$ such that $1 \in F$, $0 \notin F$, if $p \in F$ and $p \leq q$ then $q \in F$, and if $p, q \in F$ then $p \wedge q \in F$ Furthermore, a filter F is called an ultrafilter if it is contained in no other filter. As [6] shows, this condition is equivalent to saying that the filter has the properties that for every p, either $p \in F$ or $\neg p \in F$ and that if $p \vee q \in F$ then at least one of $p, q \in F$.

Theorem 2.3. *Every filter is contained in an ultrafilter.*

Proof. (Proof due largely to [5]). Suppose that F is a filter of $(P, \leq)$. Then define $Q = \{K | F \subseteq K\}$ where K is a filter. Then define the pair $(Q, \subseteq)$. As discussed earlier, $\subseteq$ is a relation that gives rise to a lattice so, in particular, $(Q, \subseteq)$ is a poset. Now, the sublattice $(Q, \subset)$ is also a poset and if $\cdots \subset Q_0 \subset Q_1 \subset \cdots$ is a sequence of filters (which exists, because we can define Q recursively in terms of propositional variables and the like) then it is ordered nicely and clearly $\bigcup Q_i$ is a filter and it contains all elements of the Q_i. Hence this union has a maximal element (by Zorn's Lemma) and so $\bigcup Q_i$ is an ultrafilter. □

Therefore, ultrafilters exist and capture the maximality that we would like to characterize our model of S4.

The last piece we need before getting to completeness is Stone's Representation Theorem, which will give us a way of associating topological spaces to modal logics.

Theorem 2.4. *The Lindenbaum algebra of S4, $\mathcal{S}4$, is isomorphic to the lattice of compact open subsets of* `Stone`*(S4), the space generated by sets $D(\beta)$ of ultrafilters over the wffs of $\mathcal{L}$ where $U \in D(\beta)$ if and only if $\beta \in U$.*

Proof. First we show that the set $\{D(\beta) | \beta \in \mathcal{L}\}$ is a basis for a topology. Recall that a basis for a topology is a collection $\mathcal{D}$ of subsets of some set X such that for all $U \in \mathcal{U}$ there is a $D(\beta) \in \mathcal{D}$ with $U \in B(b)$ and that if $U \in D_1$ and $U \in D_2$ there exists a D_3 such that $U \in D_3 \subseteq D_1 \cap D_2$. Now we have that $D(\alpha) \cap D(\beta) = D(\alpha \wedge \beta)$ (following from the definition of ultrafilter given above) and that if $U \in \mathcal{U}$ then there is a set $D(\beta)$ such that $U \in D(\beta)$. We see the second claim by noting that if U is empty, one such $D(b)$ is $D(p \wedge \neg p)$ and that if U is nonempty then there is a β such that $\beta \in U$, meaning that $U \in D(\beta)$. This forms a topology because $\bigcap_{i=1}^{n} D(\alpha_i) = D(\bigwedge_{i=1}^{n} \alpha_i)$ and because we allow for free unions, "bounded above" by the union of all ultrafilters of $\mathcal{S}4$. We denote the resulting topology by $\texttt{Stone}(\mathcal{S}4)$.

Now first show that $\texttt{Stone}(\mathcal{S}4)$ is Hausdorff, meaning that given two points $U, V \in \mathcal{U}$, there exist open sets $D, E \in \texttt{Stone}(\mathcal{S}4)$ such that $U \in D$ and $V \in E$ but $D \cap E = \varnothing$. Suppose we have distinct ultrafilters U and V of $\mathcal{L}$. The claim is that we can find two open sets separating them. As they are distinct, there is some $\beta \in U$ such that $\beta \notin V$. By maximality, V is an element of $D(\neg\beta)$ but $U \notin D(\neg\beta)$. Furthermore, $D(\beta) \cap D(\neg\beta) = \varnothing$ with both sets open, so we have a separation of U and V, making $\texttt{Stone}(\mathcal{S}4)$ Hausdorff.

We now wish to show that this space is compact, meaning that any open covering of $\texttt{Stone}(\mathcal{S}4)$ has a finite subcovering. As every open covering will be formed out of a union of our basis sets $D(\beta)$, we have that $\bigcup_{\beta\in(\mathcal{L}')\subseteq(\mathcal{L}} D(\beta) = \mathcal{U}$ is the general form of coverings. Now suppose that there is an open covering of $\mathcal{U}$ such that no finite subcovering $\bigcup_{\mathcal{L}'} D(\gamma)$ covers $\mathcal{U}$. Then taking an arbitrary finite set of D-sets $D(\alpha)$ for $\alpha \in \mathcal{L}'$ we have an ultrafilter V not included in the union of the D-sets. By maximality, for all $\alpha \in \mathcal{L}'$ such that $D(\alpha)$ is in our finite D-set, $\neg\alpha \in V$. Hence for all $\alpha \in \mathcal{L}'$, only one of $\alpha, \neg\alpha$ is a member of $\mathcal{L}'$ and so for some variable wff δ, $D(\delta) \notin \bigcup_{\beta\in(\mathcal{L}')\subseteq(\mathcal{L}} D(\beta)$ and so $D(\delta) \notin \mathcal{U}$, a contradiction. Therefore $\texttt{Stone}(\mathcal{S}4)$ is compact and Hausdorff.

Furthermore, it is a well-known topological result that the compact open sets of a compact Hausdorff space are its clopen sets. As $\texttt{Stone}(\mathcal{S}4)$ has clopen sets of the form $D(\alpha)$ (Because $\mathcal{U} - D(\alpha) = D(\neg\alpha)$ is open) and because these sets exhaust the clopen sets (though the proof will not presently be given), we form the set of clopen . Denote the set of clopen sets in our Stone space by $\texttt{COStone}(\mathcal{S}4)$, where the "CO" comes from "compact-open". We now form the Boolean algebra $(\texttt{COStone}(\mathcal{S}4), \subseteq)$ in order to finally exhibit the desired isomorphism.

Now, define the function $\|\cdot\| : \mathcal{S}4 \to \texttt{COStone}(\mathcal{S}4)$ with $\|\beta\| = D(\beta)$. This is a homomorphism because $\|\neg\alpha\| = D(\neg\alpha) = \{U|\neg\alpha \in U\} = \{U \mid \alpha \notin U\} = \mathcal{U} - \|\alpha\|$ and because $\|\alpha \vee \beta\| = D(\alpha\vee\beta) = \{U \mid \alpha \vee \beta \in U\} = \{U \mid \alpha \in U \text{ or } \beta \in U\} = \|\alpha\| \cup \|\beta\|$. The rest of the propositional connectives can be built out of negation and disjunction. Hence, if $\alpha \vdash \beta$ then $\|\alpha\| \subseteq \|\beta\|$, giving us a homomorphism of boolean algebras. Likewise, this map has an inverse, which sends $D(\beta)$ to β, preserving structure in the same way as above. We know this inverse exists because we are working in the Lindenbaum algebra of S4, so any possible ill-definedness of the inverse vanishes since if $\|\alpha\| \neq \|\beta\|$ then $\alpha \neq \beta$ and visa versa. We therefore have our isomorphism. □

Note that in proving the Stone Representation Theorem that we naturally got an interpretation function, giving at least some metamathematical reason to accept this function as the "right" function to talk about topological semantics. Note further that this argument works in general for *all* Boolean Algebras, so we have actually proved a far stronger result.

The proof of the version of the Stone Representation Theorem given above is admittedly and purposefully somewhat roundabout, because we could have simply cut to the algebraic notion of a Boolean Algebra of ultrafilters without mentioning the Stone Space at all- however, if we chose to do that then we would lose out on both the connections between interiority and necessity as well as (as we will see in the sequel) the ability to talk about sheaves in order to give a more natural account of first-order semantics than is given in classical semantics.

We can now state and prove a completeness theorem for S4. As we will see, almost all of the work was done above and all we do now is exploit the similarities we found between subset algebras of topological spaces with the interior operator and the derivability algebra of S4 with the necessity operator.

Definition A topological interpretation $(X, \|\cdot\|)$ of a logical system $\mathcal{T}$ is a topological space X together with a map $\|\cdot\| \ : \mathcal{L}_{\mathcal{T}} \to X$ such that all axioms of $\mathcal{T}$ are satisfied by the corresponding subset algebra of X.

Theorem 2.5. *There exists a topological model* $(X, \|\cdot\|)$ *of S4 that satisfies* $\alpha \vdash \beta$ *if and only if* $\|\alpha\| \subseteq \|\beta\|$*.*

Proof. First we need to check whether or not our interpretation function $\|\cdot\|$ commutes with our earlier defined operators int and $\Box$, meaning that $\|\Box\alpha\| = \underline{\|\alpha\|}$. As shown in the proof of Stone's Theorem, $\|\cdot\|$ is a homomorphism of Boolean Algebras. Then, using the Stone Theorem, construct the Stone Space of the image of the $\Box$ map, which has clopen basis sets $D(\Box\alpha)$. Now, applying the interior operation to $\|\alpha\|$ we obtain $\operatorname{int}\|\alpha\| = \|\Box\alpha\|$. Thus, the interpretation function commutes with our previously defined homomorphisms. Since modalized sentences are the only open sets, and since $\Box\alpha \vdash \Box\Box\alpha$ and $\Box\Box\alpha \vdash \Box\alpha$, iterated modalities are reduced to $\Box\alpha$, we have that for modalized wffs $\operatorname{int}(|\alpha\|) = \|\Box\alpha\|$, but since the non-modalized sentences are not open in the topology, we need some way to extend our interior function to them. The way in which we do this is to add a (topologically "small") non-open set of ultrafilters to $\|\Box\alpha\|$ that we will denote $\mathcal{V}$. We take $\mathcal{V}$ to satisfy that for all $V \in \mathcal{V}$, $\alpha \in V$ but that $\Box\alpha \notin V$, which exists because α and $\Box\alpha$ are not in general equivalent (and if they are, they would be modded out since we're working in the Lindenbaum Algebra). It is intuitively clear that $\mathcal{V}$ is not open, and this is for the reason that it is not equal to any $D(\Box\beta$. Furthermore, by the way $\mathcal{V}$ was defined, we have that $\operatorname{int}(|\alpha\|) = \|\Box\alpha\|$ as desired (since for every β not provably equivalent to or provable from α there exist ultrafilters such that $\alpha \in U$ but $\beta \notin U$). Now, this is a topological model of S4 because all the axioms are satisfied, meaning that if $\alpha \vdash \beta$ then $\|\alpha\| \subseteq \|\beta\|$ which follows by quick verification.

Now, we also get the other direction, meaning that if $\|\alpha\| \subseteq \|\beta\|$ then $\alpha \vdash \beta$. This follows from the property used in the preceding that if $D(\alpha) \neq D(\beta)$ then either $\alpha \nvdash \beta$ or the reverse. Therefore we have constructed a topological model of S4 that is complete and sound in the sense described by the theorem. □

2.5 Consistent Extensions of S4

The arguments and theorems from the above section can easily be generalized to consistent extensions of S4. To be precise, an extension of S4 consists of adding new derivation axiom(s) to S4. We will only be interested in *consistent* extensions of S4 because, frankly, inconsistent systems are not interesting in the slightest. Now, it is an elementary result in model theory that a system is consistent if and only if the system is satisfied by a model, meaning that all its axioms and derived wffs are true in the model and that $\perp$ is false in the model. What makes many consistent extensions of S4 interesting is that many are classically *incomplete*, meaning that there exists a statement γ in the language of the system such that $\mathcal{T} \nvdash \gamma$ but γ is true in all models of $\mathcal{T}$. In topological semantics, however, we can always find a model of an extension of S4 that is complete. The jist of this is that adding new axioms to our system will impose further restrictions on our interior-like operator that are still well defined and non-vacuous, and that we can always find a suitable interpretation that commutes with $\Box$ and *int*. As such, we obtain the following theorem, whose proof is largely contained in the above Completeness theorem, with some minor modifications.

Theorem 2.6. *There exists a topological model $(X, \|\cdot\|)$ of any consistent extension $\mathcal{T}$ of S4 that satisfies $\alpha \vdash \beta$ if and only if $\|\alpha\| \subseteq \|\beta\|$.*

Proof. We consider two cases, one in which we add axioms that make α equivalent to $\Box\alpha$ for all wffs and one in which the axioms do not have this effect.

In the first case, we may add axioms strong enough to add the class of theorems $\alpha \vdash \Box\alpha$ for all $\alpha \in \mathcal{L}$, in which case we would have that α and $\Box\alpha$ would be interprovable and therefore indentified in the Lindenbaum algebra. Then the necessity homomorphism $\Box$ would simply take open sets in the Stone Space and so our interpretation function would be the canonical one constructed in the proof of the Stone Representation Theorem.

In the second case, take the Lindenbaum algebra generated by $(\mathcal{T}, \vdash)$ and then take the Stone Space with open sets of the form $D(\Box\alpha)$. Then repeat the adding-one-ultrafilter step done in the earlier completeness proof and we have our result. $\Box$

Hence, even classically incomplete extensions of S4 have complete topological models. This type of extension and its canonical construction, as we will see, is the skeleton key to create complete first-order modal systems.

3 Sheaf Semantics for First-Order S4

Having shown how to bootstrap topological semantics onto propositional S4 and its extensions, we turn now to a consideration of first-order modal semantics. The strategy in this section is to do blend the semantics of first order logic with the semantics of modal logic. To do so, I introduce sheaves in the form of étale spaces and explore their strong bond to quantifiers in first-order logical languages.

3.1 Syntax and Semantics of First-Order Logic

What distinguishes a first-order logic from a general propositional logic is the presence of quantifiers and special n-ary relations called predicates that drastically increase the expressive power of the system. With this increase in expressive power, however, the semantics become substantially more complicated. Adding modal symbols to the mix further complicates the matter, making the semantics of our first-order language even harder to pin down classically. In this section we, as we did with propositional modal logics, consider the formal syntax of the first order language characterized by the derivability relation $\vdash$ and then work to find suitable topological notions that solve the problem of finding a syntax far more intuitive. We begin with a definition of a first-order language and then proceed to give a description of first-order S4.

Definition A first-order language $\mathcal{L}$ is a propositional language along with sets of variables x_i, constants c_j, function symbols f_l, n-ary relation symbols F_k and special quantificational symbols $\forall$ and $\exists$ (though we will never need to mention the latter) with wffs defined recursively by the following rules:

$F_k[x_1, \cdots, x_k]$ is a wff, all propositional wffs are first order wffs, if α, β are wffs then so is $\alpha \vee \beta$, if α is a wff then so are $\neg\alpha$ and $(\forall x_j)\ \alpha$.

A first-order *modal* language is one in which if α is a wff then so is $\Box\alpha$. Now we define the first-order logic in terms of derivability.

Definition A first order logic is a logic satisfying all of the propositional axioms in addition to the following: $(\forall x_j)\ \alpha \vdash \alpha(c_k/x_j)$ and if $\alpha \vdash \beta$ then $\alpha \vdash (\forall x_j)\ \beta$ if x_j is free in β (that is that x_j does not occur in β).

Likewise, first-order S4 is defined as the system satisfying all both the axioms of first order logic and the axioms of S4. Now, while the proofs of these theorems are outside the scope of this paper, they are supremely important in our proof of first-order S4's completeness. They are:

Theorem 3.1. *(Gödel's Completeness Theorem) Given a consistent theory $\mathcal{T}$ that extends first order logic, there exists a nonempty class of models $\mathcal{M}$ such that $\mathcal{T} \vdash \alpha$ if and only if $M \vDash \alpha$ for all $M \in \mathcal{M}$.*

and

Theorem 3.2. *(The Downward Löwenheim-Skolem Theorem) If a first-order theory $\mathcal{T}$ is consistent, then there is a κ such that for every cardinal ι less than or equal to κ, $\mathcal{T}$ is satisfiable in a model with a domain of cardinality ι.*

3.2 Sheaves over Topological Spaces

To discuss the (topological) semantics of first order logics at all, we need the concepts of both a continuous mapping and homeomorphism, which up until now have had surprisingly little application to our investigation of logic. These notions will lead us to the idea of a *sheaf*, which gives us a nice way of the semantics of quantified propositions. To begin with, a review of homeomorphisms between topological spaces is in order.

Definition A function $f : X \to Y$ is a homeomorphism means that f is bijective and that both f and f^{-1} are continuous maps.

Before we embark on our discussion of sheaves, we require one further refinement of the notion of homeomorphism, that of the local homeomorphism

Definition A local homeomorphism is a function $f : X \to Y$ such that for every $x \in X$ there is an open set $U \in \mathcal{O}(X)$ such that f restricted to U is a homeomorphism and its image $f(U)$ is open.

The motivation behind talking about the weaker notion of a local homeomorphism is so that we can exploit the local symmetries between the Stone spaces extended to include *variable-valued* formulas that still end up locally preserving their subset relations when we pass through nicely behaving local homeomorphisms. We now define the concept of a sheaf over a topological space, which is a construction centered on local homeomorphisms.

Definition A sheaf (over X) is a topological space Q such that there is a local homomorphism $\pi : Q \to X$ called the projection of F on X.

Furthermore, each sheaf admits a disjoint set of objects called fibers by the following way: given any $x \in X$ let $Q_x = \pi^{-1}(x)$, where Q_x is the called the fiber of F at x. It is then clear that the underlying set of F can be recovered by taking the disjoint union $\coprod_{x\in X} Q_x$. This feature of sheaves is intuitively captured by the following description given by Mac Lane and Moerdijk in [6] where they write that sheaves "can be pictured as a serving of shishkebab: through each point of the fiber there is a horizontal disc on which the projection π is a homeomorphism to a disc in $\mathbb{R}^2$. The discs at different points of the fiber (pieces of lamb or onion, say) may come in very different sizes. All these different servings over different points $x \in X$ are 'glued together' by the topology of F."

Now, we wish to establish a relationship between quantifiers and projections in order to see why sheaves make sense to give first-order semantics. With χ being our set of "possible values" for constants and variables of $\mathcal{L}$ to assume, we interpret all the rest of the symbols and formulae as certain subsets of χ^n for some n. In particular, to each k-ary relation R_j we associate it with a subset $\|R_j\| \subseteq \chi^k$, to each function symbol f_j a function $\|f_j\| : \chi^n \to \chi$, and a value $\|c_k\| \in \chi$ as well as giving an interpretation to formulas with free variables $\alpha[\overline{x}]$ by associating it to the subset $\left\{\overline{b} \in \chi^n \text{ such that } M \vDash \alpha[\overline{a}]\right\}$. Connective rules such as $\|(\alpha \wedge \beta)[\overline{x}]\| = \|\alpha[\overline{x}]\| \cap \|\beta[\overline{x}]\|$ hold in this interpretation (as does the complementation result), so we already have the propositional interpretation properties holding under this. Of particular note, however, is how the existential quantifier works under this type of interpretation. As [1] notes, $\|(\exists y)\alpha[\overline{x}, y]\| = \left\{\overline{b} \in \chi^n \mid (\overline{b}, c) \in \|\alpha[\overline{x}, y]\| \text{ for some } d \in \chi\right\}$. Now, if we let $\pi_n : \chi^{n+1} \to \chi^n$ send $(\overline{b}, d)$ to $(\overline{b})$ then the semantic content of the existential statement is simply the projection $\pi_n(\|\alpha[\overline{x}, y]\|)$.

Now there is a sense in which we have constructed a topological semantics for bot propositional modal logic and for first order logic, and the problem still remains as to how to formalize this link in order to come up with a suitable first-order modal semantics. To do so we simply extend the propositional semantics one propositional-modal wff at a time to fibers that stand as existential propositions and then taking the union of these fibers. We define $\|\alpha[\overline{x}]\|_\beta$, $\beta \in \mathcal{L}_{S4}$ to be the "fiberwise interpretation" whose value is all satisfying values of $\overline{x}$ lying in the projection of α on β. Then we define the full interpretation of a first-order modal wff to be $\|\alpha[\overline{x}]\| = \coprod_{\beta\in\mathcal{L}_{S4}} \|\alpha[\overline{x}]\|_\beta$.

3.3 Topological Completeness of First-Order S4 Logic

With the machinery of first-order semantics as well as the topological notions of Stone space and sheaf under our belt, we're ready to embark on the main result of this paper, the completeness of first order S4.

Theorem 3.3. *There exists a sheaf model $Q \xrightarrow{\pi} (X, \|\cdot\|)$ of First Order S4 satisfying $\alpha \vdash \beta$ iff $\|\alpha[x_1, \cdots, x_i]\| \subseteq \|\beta[x_1, \cdots, x_j]\|$*

Proof. (Much of this argument is described in [1]) The general idea of the proof is to take a class $\mathcal{M}$ of models of an extended language of regular First-order logic with modality treated as a set of relation symbol, apply G ödel's Completeness theorem to this set to get reduce it down to a manageable size by applying the L öwenheim-Skolem theorem, toplogize the resulting set after adding a large amount of constants to our language so that existential quantifications being satisfied guarantees the actual existence of a satisfying constant, and then topologizing our reduced $\overline{\mathcal{M}}$ so that it behaves as a sheaf. The reason that this method gives us a better semantics for first-order modal logics is that in passing to the sheaf space, we essentially "forget" that our "modal operator" was really a huge set of relations. As such, we reclaim the modal character of our logic from its relatively contrived classical construction in such a way that retains the completeness results of said construction.

To begin with, we somewhat backhandedly approximate the modal operator (which is a function on wffs) by adding n-ary relations $\Box_n\alpha$ for each α with n free variables. That is, we extend our language $\mathcal{L}$ to $\mathcal{L} \cup \{\Box\alpha\}$ that satisfy the modal-type axioms of FOS4. The advantage of considering this sort of language is because the resulting first-order theory satisfies the hypotheses of Gödel's Completeness Theorem and therefore admits a *nonempty* class of models $\mathcal{M}$ such that $S4 \vdash \alpha$ if and only if $M(S4) \vDash \alpha$ for every model $M \in \mathcal{M}$. This is a good first step, however, our end goal is to translate this into an explicitly topological result using only one modal operator $\Box$ rather than a large collection of first-order relations $\Box_n\alpha$. Now, by Löwenheim-Skolem, there is some cardinal κ such that the set of models such that $\overline{\mathcal{M}} = \{\text{card}(M) \leq \kappa\}$ are still complete in the classical sense just mentioned above.

Now we begin the explicitly topological construction, wherein we topologize the above set of models to form a sheaf structure satisfying the topological completeness statement $\alpha \vdash \beta$ if and only if $\|\alpha[x_1, \cdots, x_i]\| \subseteq \|\beta[x_1, \cdots, x_j]\|$. To do so, we want there to exist as many constants as there are elements in the sets of our models so that our projections from models onto our complete set $\mathcal{M}$ guarantee that if $(\exists x_i)\alpha[x_h, \cdots, x_i]$ is satisfied then so is $\alpha[x_h, \cdots, c_i]$ for some constant c_i. As such, we extend our original language $\mathcal{L}$ to $\hat{\mathcal{L}} = \mathcal{L} \cup \{c_k \parallel k \leq \kappa\}$ and then extending it further by adding (as we did before) all formulas of the form $\Box_n\alpha$, our pseudo-modal operator, a language we will denote by $\tilde{\hat{\mathcal{L}}}$. Now, following the construction given in [1], let $\mathcal{N} = \Big\{M; f_l \mid M \in \overline{\mathcal{M}} \text{ and } f : \kappa \to \text{dom}(M) \text{ is onto}\Big\}$. The function f is further specified to satisfy that $f_l(k) = c_k$ under each M_{f_l}. What we are really doing in this construction is taking these constants that we just added to our language and then for each onto function f mapping onto the domain of M giving it canonical fibers in order to make the construction of a complete sheaf easy.

We now construct our sheaf in such a way so as to incorporate the result of the Stone theorem by considering Stone Space of the Lindenbaum Algebra of first-order S4, $\mathcal{S}4$ as the germs of our new space and then adjoining the quantified sentences as our fibers. Though the precise argument won't be given here, this new space is strongly complete in that $\alpha \vdash \beta$ iff $\|\alpha[x_1, \cdots, x_i]\| \subseteq \|\beta[x_1, \cdots, x_j]\|$, as desired. $\Box$

As was the case with propositional S4, this argument can be generalized to arbitrary consistent extension of first-order S4, including the logics that are classically incomplete. Furthermore, the topological semantics given to both first-order and propositonal modal systems characterized the notion of maximality in a way unparalleled by Kripke semantics through its heavy use of ultrafilter models to form Stone spaces.

4 Conclusion

In short, we've shown using the language of sheaves and topological spaces that many modal systems that were incomplete in all their models in classical Kripke semantics. But in order to get these results, a price did have to be paid- we lost the power of Kripke semantics to easily and categorically classify modal logics according to the type of models that modeled them in a relatively straightforward fashion. It is not that this can't be done in topological semantics, but rather that it's not conceptually obvious how this is done. As such, this area is wide open for research and further development. On a more reflective note, it is certainly quite surprising from the vantage point of the set-theoretic formulation of mathematics that such a deep connection could be made between sheaves, an object of central importance in algebraic geometry, and quantifier logics on the other hand central to mathematical logic. But this connection does not become very surprising once we strip the inherited ontology of each and recontextualize them through their algebraic qualities. After trudging through this seemingly unmotivated realm of formalism, we are able to forge new bonds between mathematical objects that did not appear to have any beforehand.

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