Continuous-Time Model of Insider Trading

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1 Introduction

The stock market is one of the most financially profitable and risky of world businesses. In order to accumulate capital for business ventures, companies sell stock, and markets have been created so that stocks can be freely traded by consumers looking to increase their wealth or buffer their retirement funds. The flow of information, along with a business' balance sheets and consumer optimism, greatly influences the trade of stocks and the prices which they can draw in markets.

I will look at one type of trader specifically, the **Inside Trader**, and look to see how he/she makes his decisions and how these decisions affect the price and output of a given market. Our model is an extension of the Kyle model in *Continuous Auctions and Insider Trading* (1985) which delves into the subject by taking a continuous approach to insider trading. To extend this model we will remove the discrete details and add stochastic calculus.

First, I will give a brief overview of some major definitions in stochastic calculus, and then I will provide information on how the traders are able to buy and sell stocks in markets. From there I will create the continuous time model of insider trading with the intent of showing that there exists a equilibrium in which the pricing rule of market makers (*i*) is a smooth, strictly monotone function of the cumulative order, (*ii*) that satisfies a certain finite-variance condition, and (*iii*) that satisfies the Bellman Equation which characterizes the inside trader's optimum strategy [1].

2 Terminology

Many topics, in this specific case the movement of stock prices, that involve "randomized" outcomes are related to the study of **Stochastic Processes**. I will provide in this section an overview of some of the central definitions and theorems necessary to understand the proofs later in the model. For further reference to the principles of stochastic calculus I refer the textbook by Shreve and Karatzas (1988).

2.1 Stochastic Processes

Definition. A **Stochastic Process** is a model for a random occurrence. To deal with the random factor, a measurable space $(\Omega, \mathcal{F})$ where probability measure can be assigned is introduced. This will be called the *sample space*. The processes itself is a collection of random variables $X = \{X_t : 0 \leq t < \infty\}$ on $(\Omega, \mathcal{F})$. These random variables will take values based on a second measurable space, $(S, \mathcal{P})$, called the *state space* [3].

Definition. When the probability of an event is 1, it is said that the event will **almost surely** occur. Another way to interpret this is the probability of an event occurring tends to 1 given some limit [3].

Definition. A **filtration** is a nondecreasing family $\mathcal{F}_t : t \geq 0$ of sub-σ fields of $\mathcal{F} : \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$.

Definition. Let $X_0, \ldots, X_n$ is a sequence of random variables, and let $\mathcal{F}_n$ denotes the information contained in $X_0, \ldots, X_n$ with $E(|X_i|) < \infty$, then the sequence $X_0, \ldots, X_n$ is a **Martingale** [5] with respect to $\mathcal{F}_n$ if:

- $X_0, \ldots, X_n$ are measurable sets for all X_i
- for each $m < n$: $E[X_n|\mathcal{F}_m] = X_m$

Definition. X is a **Semimartingale** [5] if it is a real valued process defined on a filtered probability space $(\Omega, F, (F_t)_t \geq 0, P)$ that can be written as $X_t = M_t + A_t$ where M is a local martingale and A_t is a locally bounded variation.

Definition. The **Expected Value** [5] of a random variable, X, is the weighted average of all possible values of X. Formally, if X is a random variable on the probability space (Ω, Σ, P), the expected value of X, denoted $E[X]$ is the lesbesgue integral

$$E[X] = \int_\Omega X dP = \int_\Omega X(\omega) P(d\omega)$$

Definition. A **Brownian Motion** [5] is a stochastic process, X_t, with real number values such that:

1. $X_0 = 0$
2. For any $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n$ the random variables $X_{t_1} - X_{s_1}, \ldots, X_{t_n} - X_{s_n}$ are independent
3. For any $s < t$ the random variable $X_t - X_s$ has a normal distribution with mean 0 and variance $(t - s)\sigma^2$
4. The paths are continuous, i.e.,the function $t \longmapsto X_t$ is a continuous function of t

Theorem (Ito's Formula). *If f is a function with two continuous derivatives, and W_t is a standard Brownian motion [5],*

$$f(W_t) - f(W_0) = \int_0^t f'(W_s) dW_s + \frac{1}{2} \int_0^t f''(W_s) ds$$

Definition. The **quadratic variation** [3] is

$$\lim_{n\to\infty} Q_t^{(n)},$$

where,

$$Q_t^{(n)} = G(W_{\frac{j}{n}t})[W_{\frac{j+1}{n}t} - W_{\frac{j}{n}t}]^2,$$

which is the second term in the taylor approximation of Ito's Formula. For our purposes this will be denoted as $[G, W]$.

2.2 Optimization

Definition (Bellman Equation)**.** We will use the Bellman equation in this model to show that if an equilibrium pricing rule $H(y,t)$ (all of which will be defined further in the paper) satisfies the Bellman Equation then the pricing rule is both unique and optimal. The equation is formally written as:

$$\max_{\theta\in\Re}\{J_t + J_y\theta + \frac{1}{2}\sigma^2 J_{yy} + (v-H)\theta\} = 0$$

We will interpret the Bellman Equation as saying that the instantaneous profit is exactly offset by the expected change in J (defined in Lemma 1) when an optimal strategy is followed by the inside trader, and instantaneous profit is not sufficient to exactly offset J when a "suboptimal" strategy is chosen by the trader [1].

3 Market Makers and Insider Trading

Corporations and public companies alike seek to accumulate capital for their business ventures by selling a percentage of their company, in the form of **stocks**, and the people of the world are allowed to trade these stocks amongst themselves, causing the price of a given stock of fluctuate. To prevent people with information not readily available to the public, **insiders**, from using their private information to acquire large profits, **insider trading** is a federal felony. We wish to devise a model for how people who have an advantage because of a discrepancy in information, due to a strategy, to maximize their wealth.

The trade of a stock is a substantial process, which we will simplify. A **Market Maker**, such as J.P. Morgan, buys and sells stocks from companies so that consumers may readily buy and sell stocks amongst themselves. In this model we will assume that the Market Maker is risk-neutral so that they will set their prices **competitively**, meaning that they can't manipulate the price so as to maximize their own profits.

We will consider two types of traders: informed and noise traders. We will consider a single informed trader, who will anticipate the effect of his orders on the market rationally. The noise traders will be the uniformed public, and will be indifferent from the inside trader, so that his signal, which signal can not be deciphered by inverting the price function. Thus, the market will be represented by the **semi-strong efficient-market hypothesis** which states that prices reflect all publicly available information and prices change instantly to reflect changes in public information.

The informed trader will be able to infer the flow of noise trades by simply monitoring the price of a stock continuously. We will also treat his as if he were on the floor of the New York Stock Exchange and accepting orders as they arrive, instead of placing them with a specialist. This means that for every time the informed buyer purchases a stock, the noise traders must sell a stock, and there are no other workings in the market.

So why do Market Makers do this? The Market Makers make money on the bid-ask spread, playing off the differences between what people will buy an asset for and what others are willing to sell it for. In an insider trading situation, Market Makers will make their money from the noise traders, who will always constantly be on the wrong side of the bid-ask spread (the information provided to the inside trader makes sure that this trader will always be on the advantaged side of the bid-ask spread). This profit will more than compensate the money markets for their expected losses due to the inside traders and will insure that they will not close their doors [2].

This is all the applicable and important economic information one will use in this model.

4 Building the Model

We will have a time variable $t \in [0, 1]$ such that $t = 0$ coincides with the present date, and $t = 1$ represents the date of a public release of information, information that the insider has at $t = 0$. We will consider the price of the stock after the release of information as the signal, $\tilde{v}$. We will denote the distribution function of this signal by F, which will be continuous on the real line. Continuity allows us to use the inverse function theorem to show thatF^{-1} is well defined on the interval (0,1). Lastly, it is of the utmost importance that the signal of information to the informed trader be finite, therefore we will also impose the condition that the second moment, $\int_{-\infty}^{\infty} \tilde{v}^2 dF < \infty$, be finite.

As mentioned before, we will have three players in the market: the market maker, the informed trader and the noise traders. The market maker must be risk neutral so that they are willing to allow the trading of a particular stock. The noise traders, on the other hand, will make individual decisions based on their own portfolios and stock holds, and this will appear as a random process Z_t, the cumulative orders of the noise traders through time t. These traders must also be price inelastic, so that there individual demand curves are extremely steep, so that a large change in the stock of the price will not largely effect the quantity demanded of that stock. Z_t is a stochastic process and will be assumed to have Brownian motion independent of $\tilde{v}$, which has mean 0 and variance σ^2 (per unit of time). The cumulative orders of the insider through time t will be denoted X_t. Therefore, the cumulative orders of all traders Y_t is such that $Y_t = X_t + Z_t$.

The market makers can only observe the process Y, therefore insider orders and noise orders are indistinguishable, so price at any time t will only depend on Y_t. We will define the price function as:

$$P_t = H(Y_t, t), \tag{1}$$

where H is C^2 in y, continuous in $t \in [0, 1]$, continuously differentiable in $t \in (0, 1)$ and satisfies $E[H(Z_1, t)^2] < \infty$.

Let $\mathbb{F} \equiv \{\mathcal{F}_t | 0 \leq t \leq 1\}$ denote an augmentation of the increasing family of σ-fields generated by the stochastic process ξ, satisfying $\xi_0 \equiv \tilde{v}$ and

$\xi_t \equiv Z_t$ [$\forall t > 0$]. This requires that the informed trader's strategy X is adapted to $\mathbb{F}$, which allows the informed trader to know $\tilde{v}$ at the time 0 and to infer Z_t for each t.

Now, we must consider the budget constraint of the inside trader. Let $W_t = B_t + P_t X_t$, where W_t be the trader's wealth, P_t is the price of the asset, and B is the investment in the risk-free asset all at any time t. We will then define the budget constraint [1] as the following:

$$B_t + P_t X_t = B_j + P_t X_j \qquad \text{[where } j \in [0,1] \text{ and } j < t\text{]} \tag{2}$$

It follows that the change in wealth is:

$$W_t - W_j = X_j(P_t - P_j)$$

The change in investment in the risk-free asset is:

$$B_t - B_j = -P_t(X_t - X_j)$$

The most important aspect of these conditions is that the change of wealth is not dependent on the future price of the asset, therefore the inside trader is only concerned with the present.

Now, we take this formula for wealth dynamics to be applicable when we take the stochastic differential $dX(t)$ to represent the market order. We will require that the order strategy, X, be a semimartingale so that we can integrate it later using ito's formula. We shall define wealth dynamics as

$$dW_t = X_{t-} dP_t \tag{3}$$

where X_{t-} denotes $\lim_{s \uparrow t} X_s$. Since H is assumed to be smooth, $P_t = H(Y_t, t)$ is a semimartingale. We must also note that because X_{t-} is predictable and locally bounded, so $\int X_{t-} dP_t$ exists.

Since we have defined X to be as a semimartingale we are able to write $X = D - S + M$, where D and M are positive, increasing, right continuous processes and M is a martingale. The right-continuity is a **normalization**, assuring that we are taking X_t to include any jump $\Delta X_t \equiv X_t - X_{t-1}$ made at time t. When $M = 0$, $X = D - S$ can be interpreted as the difference of purchase and sales.

When the information at $t = 1$ is released there may be a jump in the price. We will not stray away from this jump so we will define the final wealth of the inside trader as

$$W_1 = (\tilde{v} - P_1)X_1 + \int_{[0,1]} X_{t-} dP_t \tag{4}$$

The bounds of integration $[0, 1]$ are necessary due to the possibility of jumps discontinuities. Integrate by parts to get the equivalent formula

$$W_1 = \int_{[0,1]} (\tilde{v} - P_{t-}) dX_t - [P, X]_1 \tag{5}$$

where [P,X] is the *optional quadratic variation* process as defined in section 2.1. We will denote the differential of this process $dPdX$. When in equilibrium $[P,X] \equiv 0$, so the formula for total wealth will simplify to

$$W_1 = \int_{[0,1]} (\tilde{v} - P_t) dX_t \tag{6}$$

This formula can be interpreted as the value of the final position ($\tilde{v}X_1$) minus the cost of acquiring it ($\int_0^1 P_t dX_t$), and happens to be the same as the formula for the perfectly discriminating monopsonist, a situation where there is a single buyer who has full market power so he/she never has to pay more than the minimum price that a company will sell a good for. This is is analogous to the situation with our inside trader.

We must now restrict the behavior of the informed trader. We must reject any and all **doubling strategies** by the informed trader or else the model will not work. A doubling strategy is when one repeatedly doubles the previous bet until a game is won, or in this case, buy and sell an asset repeatedly in the hope that noise traders will buy and sell to artificially drive up prices. To prevent this we must take the constraint

$$E[\int_0^1 H(X_{t-} + Z_t, t)^2]dt < \infty \tag{7}$$

This guarantees that the process $\int_0^t P_{s-} dZ_s$ is a martingale and a consequence of this is that liquidity traders will not lose money on average if they could always trade at the midpoint of the spread.

Finally, we must define a few terms that will be seen later in the theorems we will prove.

Definition. A **Pricing Rule** [1] is an element of $\mathcal{H}$, where $\mathcal{H}$ denotes the class of continuous functions $H : \Re \times [0,1] \to \Re$ that are C^2 in y and continuously differentiable in t on $\Re \times (0,1)$ for which $H(\cdot, t)$ is strictly monotone for each $t \in [0,1]$ and

$$E[H(Z_1, 1)^2] < \infty \qquad \text{and} \qquad E[\int_0^1 H(Z_t, t)^2 dt] < \infty$$

Definition. A **Trading Strategy** [1] is an element of $\mathcal{X}$, where $\mathcal{X}$ is the class of semimartingales X adapted to $\mathbb{F}$ such that

$$(\forall H \in \mathcal{H}) \qquad E[\int_0^1 H(X_{t-} + Z_t, t)^2]dt < \infty \tag{8}$$

The continuity of each $H \in \mathcal{H}$ implies that the above condition leaves the density function of $X_{t-} + Z_t$ on any bounded set completely unrestricted. Therefore, a sufficient condition for X to satisfy the previous condition is for the ratio of the density function of $X_{t-} + Z$ to the density function of Z_t, is bounded uniformly in t on $(-\infty, -n) \bigcup (n, \infty)$ for some $n \in \mathbb{R}$.

Definition. Given trading strategy X, a pricing rule is **Rational** [1] if it satisfies

$$H(Y_t, t) = E[\tilde{v}|(Y_s)_{s\leq t}] \tag{9}$$

This can simply be interpreted as saying that a trading strategy is rational if the price at time t is equal to the expected price derived from the information available to the inside trader given all noise trading that has happened up to that point in time.

Definition. Given a pricing rule H, a trading strategy is **Optimal** [1] if it maximizes

$$E\{\int_{[0,1]} (\tilde{v} - P_{t-})dX_t - [P, X]_1\} \tag{10}$$

Definition. An **Equilibrium** [1] is a pair (H, X) such that H is a rational pricing rule, given X, and X is an optimal trading strategy, given H.

Definition. If (H, X) is an equilibrium for any trading strategy X, then H is an **Equilibrium Pricing Rule** [1].

5 Financial Market Equilibrium

This entire model culminates with three main results. The first is to show that the (H, X) consisting of the price function and the inside order function are in equilibrium. If they are an equilibrium, then we try to find a smooth function J that satisfies the Bellman Equation and its boundary value conditions showing that $H(Y, t)$ is unique. If we can find this function then we can conclude that X are optimal given the available information.

We will be using five lemmas that I have listed in detail in the appendix of this paper (proofs have been omitted). Lemma 1 shows the construction of a solution J to the Bellman equation. The second lemma characterizes the optima for the informed trader given a pricing rule. Lemma 3 characterizes the distribution of Y, given the strategy for the informed trader, while Lemma 4 shows that the price function would be a martingale if the informed trader did not trade. Finally, the fifth lemma shows that the order process must be a martingale in equilibrium. These five lemmas will be used extensively in the proofs of the following three theorems. A few finer details about each of these lemmas are placed in the appendix for reference [1].

5.1 Determining an Equilibrium

Theorem. *If*

$$H(y, t) = Eh(y + Z_1 - Z_t) \text{ [where } h = F^{-1} \circ N] \tag{11}$$

and for each $v \in V$,

$$X_t = (1 - t)\int_0^t \frac{h^{-1}(v) - Z_s}{(1 - s)^2} ds \tag{12}$$

then (H,X) is an equilibrium.

Proof. In order to prove an equilibrium we must show that the pricing rule is rational, given the cumulative orders of the trader, and the trading strategy is optimal, given the pricing rule. To begin will prove that $H \in \mathcal{H}$ and $X \in \mathcal{X}$. This will show that $X$is a trading strategy and H is a pricing rule by definition. We simply know that $X \in \mathcal{X}$ due to the fact that the unconditional distribution of $X_t + Z_t$ is the same as the unconditional distribution of Z_t. For the other condition, we know that H is a smooth and strictly monotone function. So the process $H(Z_t, t)$ is a martingale and $H(Z_1, 1)$ has an identical distribution as $\tilde{v}$. This being the case we get the inequality:

$$E[H(Z_t, t)^2] \leq E[H(Z_1, 1)^2] = E(\tilde{v}) < \infty$$

Now we know for certain that H and X satisfy the conditions for $\mathcal{H}$ and $\mathcal{X}$.

Second, to show the rationality of the price rule H given X we will look at the conditional expectations at time t given the market markers information (the filtration generated by Y), denoted $E^I[\cdot]$, and given the insider trader's information (the filtration $\mathbb{F}$), denoted by $E^M[\cdot]$. We will rewrite the pricing rule as:

$$H(y, t) = E^I[H(Z_1, 1)|Z_t = y] \text{ where } H(\cdot, 1) = F^{-1}(N(\cdot))$$

$$N \text{ denotes the normal } (0, \sigma^2) \text{ distribution function}$$

We will use lemma 3 because it shows that the distribution of Z with respect to the inside trader's information is equal to the distribution of Y with respect to the market maker's information. Symbolically this is represented by:

$$H(y, t) = E^M[H(Y_1, 1|Y_t = y)]$$

By the markov property we get the equality $\{(Y_t = y)\} \equiv \{(Y_s)_{s \leq t}\}$ so the above becomes:

$$H(y, t) = E^M[H(Y_1, 1|(Y_s)_{s \leq t}]$$

Lemma 3 also provides us with the fact that $H(Y_1, 1) = \tilde{v}$ therefore:

$$H(y, t) = E^I[\tilde{v}|Z_t = y] = E^M[\tilde{v}|(Y_s)_{s \leq t}]$$

This final equality proves that the pricing function is rational, therefore the equilibrium (H,X) exists.

□

5.2 Determining Uniqueness

Theorem. *The pricing rule (1) is the unique equilibrium pricing rule H for which there exists a nonnegative, smooth function $J(v, y, t)$ on $V \times \Re \times [0, 1]$ satisfying the Bellman Equation:*

$$\max_{\theta \in \Re}(J_t + J_y\theta + \frac{1}{2}\sigma^2 J_{yy} + (v - H)\theta) = 0 \tag{13}$$

and the boundary condition

$$J(v, y, 1) > J(v, h^{-1}, 1) = 0 \qquad [\forall v \in V, \forall y \notin h^{-1}(v)] \tag{14}$$

where $h(\cdot) = h(\cdot, 1)$

In the appendix of this paper, Lemma 1 shows that there exists a solution to the Bellman Equation and its boundary conditions where the equilibrium pricing rule is used. This is a very important point, but I have not provided any proof in the appendix, so below is a quick sketch of the proof of the lemma.

Fix a $v \in V$. We know that $J(\cdot, 1) = j(\cdot)$ is continuous, nonnegative, and satisfies condition (14). J is a C^2 function in y and a C^1 function in t on $\Re \times (0, 1)$. Take the function

$$J(v, y, t) = E[j(v, y + Z_1 - Z_t)],$$

and differentiate in terms of y on the left side and under the expectation operator (I'm omitting the conditions that must be satisfied in order to do this). We get

$$J_y(y, t) = E[j_y, (y + Z_1 - Z)t)] = E[h(y + Z_1 - Z_t)] - v = H(y, t) - v$$

This holds for all $(v, y, t) \in V \times \Re \times (0, 1)$. Continuity of J and J_y at $t = 1$ come from the properties of the Martingale and we are left with the solution:

$$J(v, y, t) = E[J(v, y + Z_s - Z_t, s)] \qquad [\forall (v, y) \in V \times \Re, \forall\ 0 < t < s \leq 1],$$

which is our solution to the Bellman Equation.

Proof. The first step in the proof 2 or 3 is to acknowledge the "unbiasedness property," which states that the informed trader's expected price change is zero when he does not trade. This is surely consistent with the interpretation of solution of the Bellman Equation that we constructed above because the existence of a predictable component to the price change during any interval $[s, t]$ when the informed trader did not trade would render the strategy strictly optimal to trade during interval. This leads directly to the result that price changes are locally proportional to order sizes in equilibrium (i.e. $dH = H_y DY$).

Now that the unbiased property has been explained and we have shown that there exists a solution to the Bellman Equation and its accompanying boundary value conditions all we have to do is prove that the pricing rule is unique.

Suppose that H is any equilibrium pricing rule such that there exists a solution to the Bellman Equation and its boundary value conditions. The Martingale property of $H(Z_t, t)$ (described in Lemma 4) implies that

$$H(y, t) = E[h(y + Z_1 - Z_t)] \text{ where } h(\cdot) = H(\cdot, 1)$$

As shown in Lemma 2, in equilibrium $h(Y_1) = \tilde{v}$ almost surely, so $Y_1 = h^{-1}(\tilde{v})$ almost surely. So for any arbitrary scalar a, given the market maker's information at time $t = 0$, the probability that $Y_1 \leq a$ is $F(h(a))$. By Lemma 5, the distribution function of Y_1, given the market maker's information at time $t = 0$, is N. Therefore $N = f \circ h$, implying that $h = F^{-1} \circ N$ a unique function. $\square$

5.3 Determining Optimality

Theorem. *Let (H,X) be an equilibrium. Suppose H is such that there exists a smooth solution J to the Bellman equation (3) and boundary condition (4). Then*

$$dP_t = H_y(Y_t, t)dY_t, \tag{15}$$

and the process Y is distributed as a Brownian motion with zero drift and variance σ^2, given the market makers' information (i.e. the filtration generated by Y). The process $H(Z_v, t)$ is a martingale given the informed trader's information (i.e., on the filtration $\mathbf{F}$). If G has a density function and $EH_y(Z_1, 1) < \infty$, then the process $H_y(Z_v, t)$ is a martingale given the informed trader's information, and the process $H_y(Y_v, t)$ is a martingale given the market makers' information.

Proof. Lemmas 4 and 5 in the appendix give us every condition for this theorem except the fact that the price-response coefficient is a martingale. We know that $H(Y_t, t)$, a martingale on the filtration generated by Y, is equivalent to the process $H_y(Z_t, t)$, a martingale on $\mathbb{F}$, given the the equality of the distribution of Y and Z on these respective filtrations. The martingale properties of the theorem follow directly from the martingale property of $H(Z_t, t)$, written as

$$H(y, t) = E[H(Y + Z_1 - Z_t, 1)] \tag{16}$$

From here all we do is differentiate the two sides of the equation. To do this denote $H(\cdot, 1) = h(\cdot)$. From theorem 2 we have $h = F^{-1} \circ N$. Hence,

$$h_y(y) = \frac{n(y)}{f(h(y))}$$

where n is once again the normal density function $(0, \sigma^2)$, and the assumption of the theorem implies the random variable

$$\frac{n(Z_1)}{f(h(Z_1))}$$

is integrable and it follows that

$$E\left[\frac{n(y + Z_1 - Z_t)}{f(h(y + Z_1 - Z_t))}\right]$$

for almost all $y \in \Re$. For any ϵ and any $|y - y'| < \epsilon$, the random variable

$$h_y(y' + Z_1 - Z_t) = \frac{n(y' + Z_1 - Z_t)}{f(h(y' + Z_1 - Z_t))}$$

is dominated almost surely by the larger of the random variables

$$\frac{n(y \pm \epsilon + Z_1 - Z_t)}{f(h(y \pm \epsilon + Z_1 - Z_t))}$$

This follows from the montonicity of h. Therefore, the random variables for $|y - y'| < \epsilon$ are uniformly integrable. This implies that we can interchange differentiation and expectation. Therefore, we arrive at our desired result of

$$dP_t = H_y(Y_t, t)dY_t.$$

□

6 Extensions to More Complex Models

The main purpose of this model was to show that there is an optimal strategy for an inside trader who knows precisely when the information will be released. However, the author seems to have implied that the model is based around the assumption that the information that will be released is of a good nature. That is, the informed trader is using the basic strategy of the stock market: buy low and sell high. There is however a second major strategy that is not included in this model, short selling. This involves borrowing shares of stock from a market maker, normally against one's private assets, with the intent of selling of immediately selling them then buying the same quantity of shares at a lower price at later date and returning them back to a market maker. If the price of stock goes down,the trader will make money, and if the price goes up he will lose money. The main piece of the model that will have change is the effect of the inside trader on the stock price. Since he is "borrowing" the stock, their is no direct money transfers, and the stock price will not be affected by him. Instead one would have to change the optimization condition so that the trader will optimize his profit based on the fluctuations of the noise traders alone. In giving up the "pricing authority" of the inside trader in this specific situation, we will make the model even more effective; however, adapting the model to meet these standards is outside of the scope of this paper.

7 Conclusion

In this paper, Kerry Back elucidates the power that inside trader's have maximize they're welfare using their special information. By taking the Albert Kyle's model from *Continuous Auctions and Insider Trading* (1985), which happens to use discrete measures quite often, and adding stochastic calculus Back is able to recreate the model using continuous random processes. The benefit of using continuous-time models, instead of discrete time models, is that it allows the informed trader to infer the flow of noise traders without observing them, simply monitoring the price fluctuations in the marketplace. This also allows us to make extremely powerful assumptions such as $dX_t = -dZ_t$.

In the first section of the model, we defined the assumptions upon which the rest of the model would rely on (the sheer number of them make it difficult to decide which are the most important). Once the groundwork was laid, we proved the three original premises of the Kyle Model, adapted to Stochastic

Calculus. We were able to prove that an equilibrium between the pricing rule and the inside trader's trading strategy. We were then able to prove if there existed a function J that satisfied the Bellman Equation for optimization that the pricing rule was a unique equilibrium pricing rule. Finally, the last of the theorems proved that if both theorem 1 and 2 are satisfied then the cumulative orders of trader's is distributed as a Brownian Motion and thus the trading strategy of the inside trader has the potential to be optimal, meaning that it creates the greatest welfare for the trader.

Ultimately, besides the theorems key to the model is that the informed trader can move along the supply curve of the market maker at will, and because there is no predetermined cost to moving up and down the curve there are many optima available to the insider trader, mainly because of the quantifiable, yet unpredictable effect noise traders will have on the supply curve.

8 Appendix

Lemma (1). *Let h be a strictly monotone function that satisfies $E|h(Z_t)| < \infty$. Suppose the pricing rule is*

$$H(y,t) = Eh(y + Z_1 - Z_t) \tag{17}$$

Define

$$j(v,y) = \int_y^{h^{-1}(v)} (v - h(x))dx \tag{18}$$

and

$$J(v,y,t) = E[j(v, y + Z_1 - Z_t)] \tag{19}$$

where we are taking the expectation over Z, regarding v to be constant. Assume $J(v,0,0) < \infty (\forall v \in V)$. Then J is a smooth solution of (13) and (14).

For this paper "smooth" will mean that for a function J on $V \times \Re \times [0,1] \forall v \in V$, $J(v,\cdot)$ and $J_y(v,\cdot)$ are continuous on $\Re \times (0,1]$, and $J_{yy}(v,\cdot)$ and $J_t(v,\cdot)$ are continuous on $\Re \times (0,1)$ [1].

Lemma (2). *Let H be an arbitrary pricing rule and suppose there exists a nonnegative, smooth solution to (13) and (14). Then for any trading strategy X, the expected profit (10) is no large than $E[J(\tilde{v},0,0)]$. Any trading strategy $X \equiv D - S + M$ which has continuous paths, for which $M \equiv 0$, and which implies $H(Y_1, 1) = \tilde{v}$ almost surely given an expected profit equal to $E[J(\tilde{v},0,0)]$ and is therefore an optimal strategy. If X is any trading strategy that includes discrete orders, or has a nonzero local martingale part, or does not imply $H(Y_1,1) = \tilde{v}$ almost surely, then the expected profit from X is strictly less than $E[J(\tilde{v},0,0)]$.*

This lemma implies that the necessary and sufficient conditions for optimality are that there are no discrete orders (which create undesired price pressure), no "local correlation" between noise traders, and no jump in price following the announcement of the prized information. Lastly, if the market is such that the information is not fully incorporated before the release of information [i.e., if $P_1 \equiv H(Y_1, 1) \neq \tilde{v}$], then profitable trades were missed by the informed trader and an equilibrium has not been attained [1].

Lemma (3)**.** *Assume the informed trader follows the strategy (12), where h is defined in Theorem 1. Then, on the filtration* $\boldsymbol{F}$*, the process* Y *is a Brownian bridge with instantaneous variance* σ^2*, terminating at* $h^{-1}(\tilde{v})$ *On the filtration generated by* Y*, the process* Y *is a Brownian motion with zero drift and instantaneous variance* σ^2*.*

Ultimately, the purpose of lemmas 2 and 3 is to show that the strategy is optimal whenever the pricing rule follows the form in theorem 1 [1].

Lemma (4)**.** *Let* H *be an arbitrary pricing rule. Assume there exists a smooth solution* J *to (14) and (15) in text. Then the process* $H(Z_t, t)$ *is a martingale on the filtration* $\mathbb{F}$*. If* $X = D - S + M$ *is any trading strategy that has continuous paths and for which* $M \equiv 0$*, then, for all t,*

$$H(Y_t, t) = H(0, 0) + \int_0^t H_y(Y_s, s) dY_s \tag{20}$$

Lemma (5)**.** *Let* (H, X) *be an equilibrium. Assume there exists a smooth solution* J *to (14) and (15). Then, on the filtration it generates, the process* Y *must be a Brownian motion with zero drift and variance* σ^2*.*

9 References

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