# Problems of the Newtonian N-Body Problem: Off to Infinity in Finite Time

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## 1 Introduction

The first complete mathematical formulation of Isaac Newton's "$n$-body problem" appeared in his publication, *Principia*, and has to do with the predicting of the motions of a group of massive objects that interact with each other gravitationally. Physically, this problem can be described as this: given only the present positions and velocities of a group of celestial bodies, predict their motions for all future time and deduce them for all past time. In the paper, *Off to Infinity in Finite Time*, by Donald G. Saari and Zhihong Xia[6], we are introduced to a less intuitive result of Newton's law of gravitation concerning the $n$-body problem after removing the possibility for collisions. In their paper, we are concerned with a problem Poincaré and Painlevé raised about a century ago: *Without collisions, could the Newtonian $n$-body problem of point masses eject a particle to infinity in finite time?*

The purpose of this paper is to provide a review of Saari and Xia's paper as well as cover the language and tools necessary to examine more closely the problems presented.

## 2 The Newtonian N-Body Problem

To understand more generally what Saari and Xia sought to show in their paper, we must first provide a more precise statement of the traditional $n$-body problem. That is:

> Consider $n$ point masses $m_1, ..., m_n$ in three-dimensional space. Suppose that the force of attraction experienced between each pair of the particles is Newtonian. Then, if the initial positions in space and initial velocities are specified for every particle at some present instant $t_0$, determine the position of each particle at every future (or past) moment of time.

The $n$-body problem itself can be reduced to a system of ordinary differential equations (found from Newton's second law of motion); however, when $n \geq 3$, the problem becomes increasingly difficult. The goal of Saari and Xia is to modify this problem to consider a system of masses that does not have collisions. Then, attempt to find a set up such that a mass may be accelerated so it may travel an infinite distance in a finite time. More subtly, the problem is to characterize the nature of "singularities" of $n$-body systems. In Saari and Xia's context, a singularity is a "time" value $t = t^*$ where analytic continuation of the solution to the differential equations fail.

## 2.1 Solving the Two-Body Problem

Consider the $n$-body problem stated above. Let $m_j$, $\mathbf{r}_j$ be, respectively, the mass and position vector of the $j$th particle, and let $r_{ij} = ||\mathbf{r}_i - \mathbf{r}_j||$, where $||...||$ is the Euclidean length of the vector. From Newton's second law of motion, $F_{net} = ma$, we have:

$$m_j \ddot{\mathbf{r}}_j = \sum_{i \neq j} \frac{m_i m_j (\mathbf{r}_i - \mathbf{r}_j)}{r_{ij}^3} \tag{2.1a}$$

$$= \frac{\partial U}{\partial \mathbf{r}_j}, j = 1, ..., n \tag{2.1b}$$

The left hand side of equation (2.1) is just $ma$ and the right hand side is $F_{net}$ which is Newton's law of gravitation and the extra power of $r$ in the denominator is there to balance the extra factor $(\mathbf{r}_i - \mathbf{r}_j)$ in the numerator that is there to specify a direction for the force. $U$ is defined to be the self-potential, the negative of the potential energy, which is:

$$U = \sum_{i<j} \frac{m_i m_j}{r_{ij}} \tag{2.2}$$

For $n = 2$, this problem has two parts, each a one-body problem:

1. Solving for the motion of the center of mass
2. Solving for the relative motions of each body

### 2.1.1 Solving for the Motion of the Center of Mass

To solve for the motion of the center of mass, we begin by examining:

$$\mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2) = m_1\ddot{\mathbf{r}}_1 \tag{2.3a}$$
$$\mathbf{F}_{21}(\mathbf{r}_1, \mathbf{r}_2) = m_2\ddot{\mathbf{r}}_2 \tag{2.3b}$$

Addition of both equations of (2.3) gives us:

$$\mathbf{F}_{12} + \mathbf{F}_{21} = \mathbf{F}_{12} - \mathbf{F}_{12} = 0$$

Where we use Newton's third law $\mathbf{F}_{12} = -\mathbf{F}_{21}$ so if we let:

$$\ddot{\mathbf{R}} = \frac{m_1\ddot{\mathbf{r}}_\mathbf{1} + m_2\ddot{\mathbf{r}}_\mathbf{2}}{m_1 + m_2}$$

Where $\mathbf{R}$ is the position of the center of mass of the system, we obtain the equation:

$$\ddot{\mathbf{R}} = 0$$

This means that, the velocity of the center of mass is constant, so the total momentum is constant (so the linear momentum of the system is conserved), so the location of the center of mass and the velocity can be determined in the two-body problem.

### 2.1.2 Solving for the Relative Motions of Each Body

The more difficult case is in solving for the vector $\mathbf{r}^* = \mathbf{r}_1 - \mathbf{r}_2$. Taking each of the equations (2.3) and dividing each by their respective mass, and then subtracting the second from the first, we get:

$$\frac{\mathbf{F}_{12}}{m_1} - \frac{\mathbf{F}_{21}}{m_2} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \ddot{\mathbf{r}}^*$$

So by applying Newton's third law again, we obtain:

$$\ddot{\mathbf{r}}^* = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}_{12}$$

$$\mu\ddot{\mathbf{r}}^* = \mathbf{F}_{12}$$

Where $\mu$ is the constant effective inertial mass appearing for two-body Newtonian mechanics:

$$\mu = \frac{1}{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)} = \frac{m_1 m_2}{m_1 + m_2}$$

Since the force in question here is gravitational, it varies according to the inverse square law with respect to $\mathbf{r}^*$ so we can solve for $\mathbf{r}^*$ and then we have a full solution of the form[1]:

$$\mathbf{r}_1(t) = \mathbf{R}(t) + \frac{m_2}{m_1 + m_2}\mathbf{r}^*(t) \tag{2.4a}$$

$$\mathbf{r}_2(t) = \mathbf{R}(t) + \frac{m_1}{m_1 + m_2}\mathbf{r}^*(t) \tag{2.4b}$$

In the case we are solving for $n \geq 3$ bodies, the solution and equations becomes significantly more complicated, but that is not the goal of this paper.

# 3 Singularities

The question, now, is how to construct a singularity. In this context, in order for there to be a singularity, we require from equation (2.1) that $r_{ij}(t)$, the distance between the $m_i$th and $m_j$th particle as a function of $t$, to become arbitrarily small as $t \to t^*$. We know that a collision between point particles would be a singularity, but this begs the question, are all singularities collisions?

Consider a situation in which there are two objects orbiting each other in such a way that the distance between them becomes arbitrarily small, but they never "collide." This means that they "flirt" with collision but never commit to actually doing so. I'm going to do away with the $\mathbf{r}^*$ notation and just let it be $\mathbf{r}$ because it is clumsy. First, Saari and Xia develop a "nice" way to describe $r_{ij} \to 0$ by letting the configuration of possibilities for singularities for the particles in question to be given by:

$$\Delta_{ij} = \{\mathbf{r} = (\mathbf{r}_1, ..., \mathbf{r}_n) \ \in (R^3)^n | \mathbf{r}_i = \mathbf{r}_j\} \text{ and } \Delta = \cup_{i<j}\Delta_{ij}$$

So $\Delta$ is the set of all possible configurations for $n$ points in $\mathbf{R}^3$ where equation (2.1) is undefined. So the oscillatory behavior of the situation just described would be one such that $\mathbf{r}(t)$ admits a sequence $\{t_i\}, t_i \to t^*$ where $\mathbf{r}(t_i)$

approaches $\Delta$ but $\mathbf{r}(t)$ does not. This means that you can find a sequence of points that approach $t^*$ where the sequence defined by $\{\mathbf{r}(t_i)\}$ approaches $\Delta$ but the function $\mathbf{r}(t)$ does not as $t \to t^*$. The reason why this isn't a possibility (at least in the $n \leq 3$-body case) is because this requires that $r_{ij}$ is bounded away from 0 (otherwise there would be a collision). More formally, the $\limsup_{t \to t^*}(r_{min}(t)) > d > 0$, so if we admit the sequence $\{t_i\}, t_i \to t^*$, then we have that all $r_{ij}(t_i) \geq d$. Since this is the case, equations (2.1) and (2.2) are always bounded above by something finite on the order of $\frac{1}{d^2}$. The velocity is then bounded by equation (2.2) because the bound on $U$, the potential energy, gives us a bound on the possible kinetic energy so we get:

$$T = \frac{1}{2}\sum_{j=1}^{n} m_j \mathbf{v}_j^2 = U + h \tag{3.1}$$

Where $T$ is the total possible kinetic energy of the system and $h$ is a constant that comes out of the idea that the amount of energy a system has is relative. This means that the solution to equations similar to equation (2.4) exist for values of $t \geq t^*$, contradicting the assumption that $t^*$ is a singularity. Because of this, we have:

**Theorem 1.** *The n-body problem has a singularity at $t = t^*$ iff*

$$\boldsymbol{r}(t) \to \Delta \ \textit{as}\ t \to t^*$$

This theorem was proved by Painlevé in a similar fashion to the sketch in the paragraph above[3]. There is some ambiguity, though, as Painlevé tells us that a singularity requires that $\mathbf{r} \to \Delta$, he doesn't state anything about whether or not that the particles must collide. In order to examine this further, we must say what we mean by a collision singularity more precisely:

**Definition 1.** *A singularity at time $t^*$ is a collision if there is $\boldsymbol{q} \in \Delta$ so that $\boldsymbol{r}(t) \to \boldsymbol{q}$ as $t \to t^*$. Otherwise, the singularity is called a noncollision singularity.*

This means that if there is a specific configuration in which $\mathbf{r}$ tends to as $t \to t^*$, then there is a collision, and otherwise, it is a noncollision singularity (i.e. the oscillatory behavior described above). Painlevé proved that for $n = 3$, all singularities are collisions, but the paper is in ascii French. My abilities to read French are limited as is, and ascii French is quite a ways out of my league.

## 3.1 Collision Singularities

To understand why that, for the $n = 3$ case singularities are always collisions, we need to relate the maximum and minimum spacing of the particles. In examining equation (2.2), we note that $U^{-1}$ can be considered a measure for $r_{min}(t)$. To find a description for the maximum distance between two particles, we can use the square root of the moment of inertia for a system of point particles, $I = \frac{1}{2}\sum_{j=1}^{n} m_j \mathbf{r}_j^2$ because if we take the center of mass to be the origin, then the maximum distance will be one of the terms of $\mathbf{r}_j^2$, so to get a bound on it, then it can be, at most, the square root of the sums of the distances. By differentiating $I$ twice and then relating that to equation (3.1), we get:

$$\ddot{I} = U + 2h \tag{3.2}$$

This specifies the intuitive relation between particles, such as when any two particles come close together, $r_{ij}$ becomes small so $U$ has a large value which, in turn, increases the acceleration of the increase of size in the moment of inertia of the $n$-body system. The extreme example is such one in which $r_{min}(t) \to 0$ (or, equivalently, $U \to \infty$). This means that $\ddot{I} \to \infty$ by equation (3.2), which is basically overkill for this proof because all we need is that $\ddot{I}$ is eventually positive because that means that $I \to A, A \in [0, \infty]$. In the case that $A = 0$, that means the moment of inertia of the 3-body system goes to zero which means they all collide at their center of mass. In the case $A \in (0, \infty]$, we use the assumption that $r_{ij} \to 0$. Since the moment of inertia of the triangle defined by the three particles tends to something $> 0$, and $r_{min}$ tends to zero, two of the sides of the triangle are bound away from zero and the other must shrink to zero. The reason they cannot switch roles in which side defines $r_{min}$, resulting in the oscillatory situation described earlier, is because the triangle inequality eventually prohibits it when $r_{min}$ becomes sufficiently small, so in the limit for $r_{min}$, two particles would then collide. Thus, for a singularity to occur, the three particles must have a collision.

## 3.2 Non-collision Singularities

After Painlevé proved that all $n = 3$ singularities were collisions, he wondered the following:

> For $n \geq 4$, can $\mathbf{r}$ approach $\Delta$ without approaching some point on the set?

The next great contributor to answering this was Edvard von Zeipel. He noted that the inverse square law imposes a negligible acceleration on particles when they are far apart so as $r_{ij}$ becomes large, the trajectories of a body becomes very much linear and came up with a surprising result:

**Theorem 2.** *A noncollision singularity occurs at time $t^*$ iff $I \to \infty$ as $t \to t^*$*

This intuitively makes little to no sense because it requires that a particle travels an infinite distance in a finite time. Remember that we are working under newtonian mechanics and not relativity theory, so there doesn't exist an upper bound on the velocity of an object, but it does require that a particle asymptotically approach the line $t = t^*$ (so there is no analytic extension of the solution of the differential equation beyond $t^*$)[7].

The behavior of these collisions were studied closely by Saari and H. Pollard where colliding particles were asserted to tend toward each other like $(t - t^*)^{-\frac{2}{3}}$. The $\frac{2}{3}$ exponent comes from the inverse $p$ force law, where the exponent is $\frac{2}{p+1}$ and it happens that Newton's law is $p = 2$. This is shown from the solution of the collinear inverse force relation equation:

$$\ddot{r} = -(p-1)r^{-p}$$

Multiplying both sides by $\dot{r}$ and then finding the antiderivative,

$$\int \ddot{r}\dot{r}dr = -(p-1)\int \dot{r}r^{-p}dr$$

Then, for the left hand side,

$$\frac{1}{2}\int (\dot{r}\dot{r})'dr = \frac{1}{2}\dot{r}^2$$

and for the right hand side,

$$u = r \qquad du = \dot{r}dr$$

$$-(p-1)\int \dot{r}r^{-p}dr = -(p-1)\int u^{-p}du = u^{1-p} + h$$

where $h$ is a constant of integration. After we resubstitute $r$ for $u$, we have,

$$\frac{1}{2}\dot{r}^2 = r^{1-p} + h$$

or,

$$\frac{1}{2}\dot{r}^2 r^{p-1} = 1 + hr^{p-1}$$

so by taking square roots, multiplying by $\sqrt{2}$, and examining the left hand side,

$$\dot{r}r^{\frac{p-1}{2}} = \sqrt{2}(1 + hr^{p-1})^{\frac{1}{2}}.$$

Which in the limit as $r_{min} \to 0$ is,

$$\dot{r}r^{\frac{p-1}{2}} \to \sqrt{2} \quad \text{as} \quad r_{min} \to 0.$$

Taking one more antiderivative yields:

$$\frac{2}{p+1} r^{\frac{p+1}{2}}$$

which is apparently where the $\frac{2}{p+1}$ comes from in the inverse $p$ force law.

The importance of this result lies in the fact that $I \sim A(t - t^*)^{-\frac{2}{3}}$ so we have, from equation (3.2)[4],

$$U = \ddot{I} + h \sim A(t - t^*)^{-\frac{2}{3}} \quad \text{as} \quad t \to t^* \tag{3.3}$$

This shows that $I$ and $\dot{I}$ are bounded, so in order to create a non-collision singularity, $\ddot{I}$ needs blow up *very* quickly by having $r_{min} \to 0$ *very* rapidly. It is hard to imagine such a universe that explodes so quickly, but by equation (3.2), we can find such relations for $I$ so that it goes to infinity faster than functions similar to $ln((t^* - t)^{-1})$ as $t \to t^*$.

Physically, the key to having this happen without a collision is that particles must approach other distant particles infinitely often and arbitrarily closely because as von Zeipel notes[7], the inverse square law has little effect on particles where $r_{ij}$ is large. This means that if we are only considering a single close-contact interaction, the particles would accelerate, but then, eventually, move at a nearly constant rate so that the prospects of traveling an infinite distance in a finite time would be impossible. To bypass this, we need another close visit with another particle, and then another, and another and so on, so that we have close visits accumulating in such a way that we can "boost" the particles acceleration to produce a singularity at $t = t^*$. An important question to be asked is: how likely is it that we can have noncollision singularities existing in a $n$-body situation? Again, we go to von Zeipel and another paper by Saari[5]:

**Theorem 3.** *The set of initial conditions leading to collisions is of first category.*

A set being of first category means that, given a space $M$, a subset $A$ of $M$ is of first category if it can be expressed as the union of countably many sets that are nowhere dense. This means that noncollision singularities for $n$-body systems are very unlikely and our goal, now, is to find some constructions that allow for them.

# 4 Shooting off to Infinity

The rest of this paper will be devoted to examining different constructions that produce noncollision singularities in the $n$-body problem.

## 4.1 The Mather-McGehee Construction

The 1975 paper by Mather and McGehee showed that for the collinear four-body problem that binary collisions, collisions between two of the bodies, can accumulate in such a way to eject particles to infinity in finite time. This didn't resolve the Painlevé problem because it requires that a noncollision singularity must be the first singularity of the system, but it hinted strongly that a configuration exists. It builds on behaviour of near-triple collision orbits for the collinear three-body problem.

For binary collisions, the dynamics mimic those of elastic collisions. A collinear triple is one such that the three bodies collide at the same time, and a near-triple collision is one in which there are two binary collisions in very close time proximity to each other. McGehee developed a form of "spherical coordinates" where the radius of the system is defined by $r_{max} = I^{\frac{1}{2}}$, so the "angular coordinates," $\frac{1}{r_{max}}(\mathbf{r}_1, ..., \mathbf{r}_n)$ configuration represents the particles. Mathematically, this system is defined even for $r_{max} = 0$, as it defines the zero point in $\Delta$. Geometrically, this means that it creates an invariant boundary manifold $C$ called the "collision manifold." The solution to the dynamical system extends smoothly to the $C$ because everywhere inside of it, we have fine, analytic solutions to the system of differential equations defining the four-body problem[2]. Since this is the case, we examine a "gradient-like" flow that occurs on $C$ and make conclusions about near-triple collision motion.

The consequences of this set-up is similar to an easy to set up experiment in which we drop an elastic ball to the ground and then another so that just as the first rebounds, the second ball collides with it and is transferred extra momentum, thus resulting in a greater final momentum than initial.

Near-triple collisions behave accordingly. Suppose we have three masses, $m_1, m_2, m_3$, such that their motions are restricted to a line with $m_2$ between $m_1$ and $m_3$. Let $m_1$ and $m_2$ collide and $m_3$ arrive just a little bit late. From equations (3.1) and (3.3) we find that the rebounding velocity between $m_1$ and $m_2$ can be arbitrarily large since $U$ is unbounded. This results in an arbitrarily large amount of momentum being exchanged into $m_3$ from $m_2$, and from the same reasoning as between $m_1$ and $m_2$, $m_3$ leaves the collision with an arbitrarily large(r) momentum.

However, because we have from equation (3.3), a bound on $I$ as well as $\dot{I}$, we want to keep $m_3$ from being ejected off into infinity, so we must add a fourth mass, $m_4$ so that with appropriate timing, $m_3$ will eventually come into contact with $m_4$ in such a way that will not cause $m_4$ to accelerate off into the abyss (which only requires that $m_3$ had slowed enough from its gravitational attraction to $m_1$ and $m_2$ before it collides with $m_4$) so that it rebounds back toward $m_1$ and $m_2$ bringing with it the same arbitrarily large momentum that it left with and repeating the cycle. Note that this also pushes $m_4$ slightly farther away, so with the appropriate choice in initial conditions, it is conceivable to find a way to cause collisions to happen at an unbounded rate, which would cause $\ddot{I}$ to become unbounded, causing a noncollision singularity the moment the final particle to join the party has an acceleration high enough to reach infinity.

## 4.2 The Xia Construction

Xia's construction is an attempt to answer Painlevé's question of whether or not we could create a configuration like Mather and McGehee except without using collisions to create a noncollision singularity. By removing the collisions of the Mather and McGehee construction, we must turn to near multiple collisions. We proceed by considering again the three-body case in which we have the motions of $m_1$ and $m_2$ always parallel to the $x$-$y$ plane and $m_3$ is restricted to the $z$-axis. In the two body problem between $m_1$ and $m_2$, we want the motion of their center of mass to be at rest on some point on the $z$-axis and their motions relative to the $z$-axis to be elliptical with the axis at one foci. For this solution to occur in the two-body system, they must travel symmetrically about their common center of mass. Then we can allow the eccentricity of their ellipses to become so large their motions become nearly linear. Then as they make close contact and *just* as they begin to pass, we allow $m_3$ to pass by along the axis which causes $m_1$ and $m_2$ to impart an amount of momentum to $m_3$ which can be arbitrarily large by allowing the distances between their near collision to become arbitrarily small.

Again, to prevent $m_3$ from shooting off into infinity, we must add another obstacle. To do so, we add two more masses, $m_4$ and $m_5$ that behave similarly to $m_1$ and $m_2$ in that they have a fixed center of mass lying on the $z$-axis and that they pass arbitrarily close to each other in highly eccentric elliptical orbits. With a perfect timing, we can allow $m_3$ to pass *just* before $m_4$ and $m_5$ have a near collision, which then can impart an arbitrarily large amount of momentum to $m_3$, launching it back to $m_1$ and $m_2$.

Similarly to the Mather and McGehee, this process can be repeated infinitely often in a finite amount of time, which then creates a noncollision singularity[6].

# 5 Conclusion

The possibility for a collisionless newtonian $n$-body system to propel an object an infinite distance in a finite amount of time is quite astonishing, as it, even without considering an upper bound to the potential speed, is counterintuitive that there could then exist a time in which we cannot resolve "where" an object went. This result, however, can only exist in theoretical newtonian mechanics because of the aforementioned "upper speed limit" in the world we live in, so one should take care in considering the possibility of "coming out of existence" in a non-conventional sense at some time $t^*$.

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