Graph Embeddings and the Robertson-Seymour Theorem

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1 Introduction

In a series of twenty papers spanning 1972-2004, Niel Roberson and P.D. Seymour published a large body of work on graph theory including a proof of Wagner's Conjecture (now known as the Graph Minor Theorem).

This paper aims to give an overview of necessary graph theory background and provide motivation for Robertson and Seymour's work. We cover embeddings in general, but focus on the understanding them in detail on the plane to build intuition for the general case considered in the Graph Minor Theorem. We examine Kuratowski's famous result characterizing planar graphs by a set of two excluded minors which naturally led mathematicians to ask if such a finite listing of forbidden minors is possible for other surfaces.

Wagner conjectured that any minor closed set of graphs can be characterized by a finite set of excluded cycles. Embeddings on a surface are minor closed. However, the projective plane is the only other surface besides the plane and the sphere where these forbidden minors are known. For non orientable surfaces, Archdeacon and Huneke established in 1980 that such a list of excluded minors is finite. But it was not until Robertson and Seymour's proved the Wagner conjecture that the result was shown for orientable surfaces. [5]

We discuss another equivalent way to look at the Roberson–Seymour Theorem in terms of well-quasi-orderings as well discuss some general concepts behind the proof. We also cover some algorithmic applications of Robertson and Seymour's work.

2 Basics of Graphs and Embeddings

Here we cover the very basic definitions and concepts used throughout the paper.

2.1 Graphs

A **finite simple graph** G is defined by two finite sets $V(G)$ and $E(G)$, where an element of $V(G)$ is called a vertex and an element of $E(G)$ is an edge. Each edge is a two element subset of vertices. An edge e from vertex u to vertex v is usually written as uv, and we say that e is **incident** with u and v.

A **multigraph** is similar to a simple graph, but loops uu and multiple edges from u to v are allowed.

A **vertex degree** of v denoted by $d(v)$ is the number of edges of G incident with v, with loops counting twice. Intuitively, this is a count of how many ends of edges meet at v.

$$\sum_{v \in V} d(v) = 2|E|$$

2.2 Connectivity

We define some necessary terms related to graph connectivity.
A **walk** in a graph G connecting vertices v_0 and v_n is a sequence

$W = v_0, e_1, v_1, e_2, \ldots, v_{l-1}, e_l, v_l$ where each edge e_k joins the vertices v_{k-1} and v_k . A **path** from v_0 to v_n is just the edges(in order) of the walk W from v_0 to v_n. A **cycle** is a path where $v_0 = v_l$.
A graph is **connected** if for any two vertices v_a and v_b, there exists a walk connecting them. If a graph is not connected it can be divided into maximal connected **components**
A **vertex cut** of a graph G is a set of vertices S such that $G - S$ becomes disconnected. A graph G is **k-connected** if there is no vertex cut of G with $k - 1$ of fewer vertices. Note that any connected graph is $1 - \textit{connected}$

2.3 Blocks

A **block** is a maximal connected subgraph that has no cut vertex (is non separable).
We would like to be able to **decompose** the graph G into subgraphs $H_1, H_2, \ldots, H_i$ such that each subgraph is 2-connected and $G = \bigcup_{n=1}^{i} H_n$. Since any two blocks of G have at most one vertex in common, we can create such a decomposition just by letting the subgraphs H_n be the blocks of G.

Here is an example of a block decomposition:

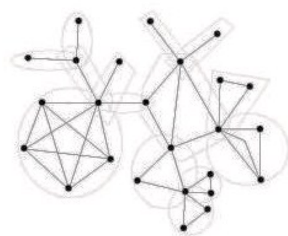


Figure 1: A decomposition into non separable subgraphs - a block decomposition

It is interesting to note that algorithmically, finding blocks is really easy. It can be simply implemented as a Depth-First-Search.

2.4 S-Components

Let G be a connected graph with a vertex cut S. If X is one of the components of $G-S$, then the subgraph H of G induced by $X \cup S$ is called an **S-component** of G.

In the case of a 2-connected G with a vertex cut $S = \{x, y\}$, we sometimes add the edge $e = xy$ to each {x,y}-component. In that case these new graphs are called **marked S-components**.

Theorem 1. *Let G be a 2-connected graph and let S be a 2-vertex cut of G. Then the marked S-components of$G$ are also 2-connected.*

This property allows us continue dividing each marked $\{x, y\}$-component into smaller marked components until each component does not have a two vertex separating set, leaving us with 3-connected graphs. We call them the **3-connected components**.

2.5 Special Graphs

We introduce and describe notation for some important special graphs.

2.5.1 Complete Graphs

A **complete graph** is a simple graph where any two vertices are connected with an edge. We denote a complete graph with n vertices K_n. Here is K_5, a graph that will be very important to our discussion of planar embeddings:

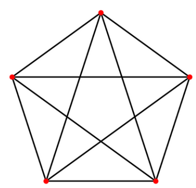


Figure 2: A possible drawing of K_5

2.5.2 Bipartite Graphs

A **bipartite graph** is a simple graph where the vertices can be partitioned into two sets A and B such that every edge has one vertex end in A and one in B. A **complete bipartite graph** has an edge from every vertex in A to every vertex in B. We denote the complete bipartite graph with p vertices in A and q vertices in B $K_{p,q}$. Here is an example of a complete bipartite graph $K_{3,3}$ which will be very important in our discussion of planarity.

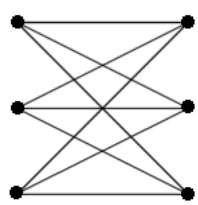


Figure 3: A possible drawing of $K_{3,3}$

2.5.3 Trees

A **tree** is a connected graph with no cycles(**acyclic**). Here are a couple easy theorems about trees to build intuition:

Theorem 2. *In a tree, any two vertices are connected by exactly one path.*

Proof. Fix two vertices: v_1 and v_2. We already know that there exists a path from v_1 and v_2. Suppose there exist two paths p_1 and p_2 and prove a contradiction. We can create the path p_2' from p_2 by reversing the edges of p_2 . We get that p_2' is a path from v_2 to v_1. So we can glue the paths the paths p_1 and p_2'. We get $P = p_1p_2'$ where P is a path from v_1 to v_1 and is therefore a cycle. We get that our tree actually contains the cycle P. Contradiction. □

Theorem 3. *For any tree* $Te(T) = v(T) - 1$

Proof. We prove this by induction on the number of vertices. If $v(T) = 1$, $e(T) = 0$ (otherwise T would have a cycle). If $v(T) \geq 2$ find a vertex v of degree one (a leaf). Remove that v and the only edge e attached to v to create T'. $v(T') = v(T) - 1$, $e(T') = e(T') - 1$, which combined with the inductive hypothesis gives us $Te(T) = v(T) - 1$. □

2.6 Embeddings

Intuitively, embedding a graph onto a surface means drawing the graph on the surface such that no edges cross. More formally, an embedding is a representation of graph G on a surface S where vertices are assigned to be points on S and edges are simple arcs connecting the points of the vertices where no two arcs may cross.

We go into detail about embeddings in the plane in the next section.

On the other hand, talking about embeddings in $\mathbb{R}^3$ is not interesting, as every graph can be embedded in $\mathbb{R}^3$. There are many ways for creating such an embedding. For example, any graph can be drawn with straight line edges if the vertices are placed onto $\mathbb{R}^3$ such that no four vertices are coplanar. Here are a couple ways to do so:

1. Place vertices on the curve $\{(\sin t, \cos t, t) | 0 \leq t \leq \pi/2\}$ [6]

2. Place the i^{th} vertex at (i, i^2, i^3)

2.7 Subdivisions

A **subdivision** replaces and edge e joining v and w by a new vertex u and edges vu and uw. Two graphs are **homeomorphic** if there is some graph from which each can be obtained by a sequence of subdivisions. Bellow is an example of three homeomorphic graphs.

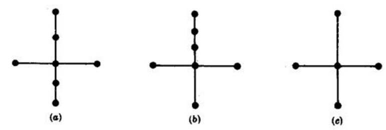


Figure 4: Three homeomorphic graphs

2.8 Minors and Contractions

A minor of a graph G is a smaller graph H that can be created from G by a series of edge contractions and deletions.

- **An edge deletion** simply creates a new graph G' where $V(G') = V(G)$ but $E(G') = E(G) - e$ where e is the deleted edge.
- **A contraction** of an edge uv replaces the vertices u and v with a new vertex w with an edge wx in G' for every edge ux and vx in G

Note: a minor is independent of the order of operations performed and therefore can simply be characterized by a list of edge contractions and deletions.

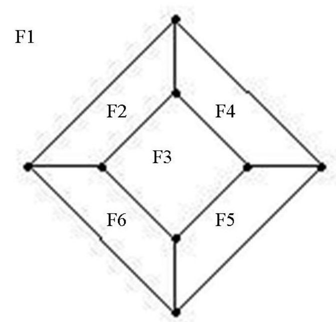
3 Planar Graphs

Here we discuss planar graphs: graphs that can be embedded onto the plane.

3.1 Definitions

A **region** of a plane embedded graph G is a maximal (arcwise) connected set of points on the plane but not in the embedding of G (in the complement of G relative to the plane). Harris, Hirst and Mossinghoff give a great visual intuition in their Combinatorics and Graph Theory: If we imagine a cookie cutter in the shape of an embedding of G, then the cookies are the regions [4]. One thing to note: each plane graph has exactly one unbounded region(a huge cookie!), usually called the **outer face** or **exterior face**. See bellow for a graph with its faces labeled. F1 is the unbounded face.

We define $b(R)$ to be the number of edges bounding the region R. In the graph bellow, each region R has $b(R) = 4$



3.2 The Jordan Curve Theorem

To talk about regions, we need some basic knowledge of topology. The most important result for graph theory is The Jordan Curve Theorem.

Theorem 4 (The Jordan Curve Theorem)**.** *Any simple closed curve C in the plane partitions the plane into exactly two disjoint arcwise connected sets. Both of these regions have C as the boundary.*

This intuitively very obvious result is not obvious to anybody who wants to be rigorous about topology. I omit the proof for the sake of brevity and focus. However, readers interested in a rigorous topological proof can refer to Graphs on Surfaces by Mohar and Thomassen.

3.3 Euler's Formula and Consequences

Euler discovered a relationship between the counts of vertices, edges and regions of embedded graphs. Euler actually proved a more general relationship, but I present the formula for the plane.

Theorem 5. *If a planar embedding of a connected graph G in the plane has n verices, m edges and r regions, then $n - m + r = 2$*

Proof. We induct on the number of edges q.

If $q = 0$, then G must be K_1 with $n = 1$ and $r = 1$ since the embedding of G only has the outer face. So Eurler's formula holds for $q = 0$.

Assume the formula is true for graphs with fewer than q edges, and G has q edges. There are two cases:

Case 1: Suppose G is a tree. Then $r = 1$, and we know that for trees $q = n - 1$. We get $n - q + r = n - (n - 1) + 1 = 2$.

Case 2: If G is not a tree, but is connected it has a cycle. Pick an edge e that is part of a cycle. Let H be G with e removed. H is still connected and has one fewer edge ($e' = e - 1$), so we can use the inductive hypothesis. Since e was part of a cycle in G it was separating two difference faces. So with e removed the two faces coalesce into just one and H has one fewer face ($r' = r - 1$). The number of vertices stayed the same. We have $n - q + r = n - (q - 1) + (r - 1) = n' - q' + r' = 2$ □

There are many other variants of this proof. However, many proofs come down to either deleting or contracting an edge in a cycle. Like we have seen, deleting an edge decreases the number of faces and edges by one. Contracting an edge decreases the number of vertices and edges by one, keeping the same number of faces.

For a more topologically rigorous proof of Euler's Formula, I again refer the reader to Mohar and Thomassen.

From Euler's formula we can create a very easy way to classify some graphs as non planar.

Theorem 6. *A planar graph G with $n \geq 3$ vertices has $q \leq 3(n-2)$ edges.*

Proof. We prove this for connected graphs. If a graph H is disconnected, add edges until it is connected, and if the inequality holds for the connected H' with more edges it certainly does for H.

We try relate q and r. Consider the sum

$$C = \sum_R b(R)$$

over all the regions of G. Since every edge of G can be at the boundary of at most two regions, $C \leq 2q$. Since G is connected, and has more than 3 vertices, each face must have at least three edges bounding each region, so $C \geq 3r$. We get $3r \leq 2q$ which combined with Euler's formula gives us $q \leq 3n - 6$ □

Corollary 7. *The complete graph of five vertices K_5 is not planar.*

Theorem 8. *A planar graph G with $n \geq 3$ vertices has $q \leq 2(n-2)$ edges if G is bipartite.*

Proof. This proof is very similar to the previous one, except bipartite graphs cannot contain a cycle with fewer than 4 edges. Therefore each region is bounded by at least 4 edges and $C \geq 4r$. We get

$$4r \leq 2q \implies 4(2 + q - n) \leq 2q \implies q \leq 2(n-2)$$

□

Corollary 9. *The full bipartite graph $K_{3,3}$ is not planar.*

3.4 Kuratowski's Theorem

From the two previous corollaries we know that K_5 and $K_{3,3}$ cannot be embedded on the plane. Are there any other graphs that cannot be embedded? Yes.

Proposition 10. *If a graph G has a subgraph that is a subdivision of $K_5 or K_{3,3}$, then G is non planar.*

Proof. Every subgraph of a planar graph is planar and subdividing edges does not effect planarity (an embedding of G can be used to get an embedding of a subdivision of G, and vice versa). □

So we found a bunch more graphs that cannot be embedded in the plane - those that contain a subdivision of $K_5 or K_{3,3}$ (a Kuratowski subdivision). Are there more? Let us consider minors instead of subgraphs.

Proposition 11. *If a graph G contains a minor of $K_5 or K_{3,3}$, then G is non planar.*

Proof. Minors of planar graphs are minors, as an embedding for a minor can be created from an embedding of the original graph. □

So did we just find a bunch more non planar graphs? Not really. If graph F is homeomorphism of a subgraph of G, then F is also a minor of G. We can obtain F as a minor, first by deleting edges not in the subgraph, and then contracting edges that were subdivided. Conversely, if G has $K_5 or K_{3,3}$ as a minor, it also has a Kuratowski subdivision.

We now know of many graphs that are not planar: those that have $K_5 or K_{3,3}$ as a minor or equivalently, those that contain a Kuratowski subdivision. Are there any more? No. This is the statement of Kuratowski's and Wagner's Theorems.

Theorem 12 (Kuratowski's Theorem)**.** *A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.*

Or equivalently.

Theorem 13 (Wagner's Theorem)**.** *A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.*

The proofs I have seen of these theorems have been similar, with slightly different details depending on whether Wagner's or Kuratowski's Theorem is proved. However (at least the ones I have seen) generally follow the basic outline bellow.

1. Prove that G is planar if and only if each block of G is planar.

2. Prove that G is planar if and only if each of the 3-connected components is planar.

3. Prove that for every simple 3-connected graph other than K_4 has an edge whose contraction results n a 3-connected graph.

4. Induct on nodes and reduce the graph by contracting an edge. Assume the reduced graph is planar, and prove that if the original graph is non planar then un-contracting that edge created either $K_{3,3}$ or K_5 as a minor.

4 Overview of the Robertson-Seymour Theorem

There are two ways of formulating the main result of Roberson and Seymour's work. One concerns ideas similar to those of "bad" minors of Wagner's Theorem. The other formulates the result in terms of order theory.

4.1 Excluded Minors

To say that a class of graphs $\mathcal{K}$ is minor-closed means that for every graph H in $\mathcal{K}$ all the minors of H are in $\mathcal{K}$ as well. The idea is to look at a family of graphs $(M_1, M_2, \ldots)$ such that every graph G not in $\mathcal{K}$ has one of the graphs M_i as a minor. This family is known as the excluded minors of $\mathcal{K}$. Obviously such a list exists - just list all the graphs not in $\mathcal{K}$. However Wagner conjectured and Roberson–Seymour proved that for any minor closed class, there exists a finite list of excluded minors. Furthermore, the minimal list of excluded minors is unique. [5]

4.2 Well-quasi-ordering

A reflexive, transitive relation $\leq$ on a set X is a **well-quasi-order (wqo)** if

1. there is no infinite list of elements $x_1, x_2, \ldots$ with $x_i \in X$ such that $x_1 \geq x_2 \geq x_3 \ldots$ (an infinite descending chain), and

2. there is no infinite set of pairwise incomparable elements (an infinite antichain)

Equivalently, a set X is wqo if given any sequence $x_1, x_2, \ldots$ of elements of X, there exists an infinite subsequence $x_{i_1}, x_{i_2}, \ldots$ where $x_{i_1} \leq x_{i_2} \leq \ldots$

We can narrow our discussion to graphs. If T is an infinite set of graphs, then it is a consequence of Ramsey's theorem for infinite graphs that at least one of the following happens [1]:

- T contains an infinite descending chain $t_1 > t_2 > t_3 \ldots$
- T contains an infinite antichain
- T contains an infinite ascending chain $t_1 < t_2 < t_3 \ldots$

Now consider the minor relation (the proper minor relation in case of $<$). There cannot be an infinite descending chain of proper minors. Therefore an infinite set of graphs either contains an antichain or an infinite ascending chain. Robertson and Seymour proved that there cannot be an infinite antichain under the proper minor relationship.

Note, that this immediately implies that the set of excluded minors must be finite. Let S be the set of minor minimal graphs not in $\mathcal{K}$. Since no two elements of S can be a minor of each other, S is an antichain and therefore finite.

5 Some Details of the Graph Minor Theorem

Robertson and Seymour were able to prove that the set of graphs is wqo under the minor relationship. Restricting our graphs to trees, we have a simpler theorem proved by Kruskal in 1960.

Theorem 14. *The set of trees is wqo under topological containment*

Here are the two main theorems that allowed Robertson and Seymour to prove the general case.

Theorem 15. *For each positive integer k, the proper minor relation is a well-quasi-order on the set of graphs having tree-width at most k.*

and

Theorem 16. *For each positive integer k, there is an integer f(k) such that every graph with tree-width of at least f(k) has a k-grid minor.*

The first theorem shows that if we are to find a infinite antichain, the elements of that chain must have unbounded tree width. That allows us to conclude from the second theorem that the elements have arbitrarily large grids. This structure forces certain further restrictions on the elements if they are to be an antichain. Eventually by forcing more an more structure on the elements, Robertson and Seymour showed that two of the elements in the antichain are actually in a minor relationship. Leading to a contradiction. [1]

6 Applications to Algorithms

6.1 Testing for a Minor Closed Property

Not only did Robertson and Seymour prove that the list of excluded minors is finite for any minor closed property, but they also provided an algorithm for testing if a fixed graph H is a minor of any input graph G. The running time of this algorithm is $O(n^3)$ (where n is the size of G). Which means that testing if G has a minor closed property is also polynomial time on the input. All that is necessary is to check if any of the finitely many excluded minors are a minor of G.

Unfortunately, although this is a really cool result theoretically, it only proves an existence of an algorithm and doesn't actually provide one. The catch is that even though we know the list of excluded minors is finite, we rarely know what that list contains. Even for properties where we do know the excluded minors (the plane, sphere, projective plane) the algorithm is quite inefficient. The inefficiencies lie in the huge constant factors, as the constant factor in minor testing grows very rapidly with size of the minor H. The good news is there exists a planarity testing algorithm that runs in $O(n)$ thanks to Hopcroft and Tarjan(1974) [3].

6.2 Tree Decompositions and Tree Width

We discuss tree decompositions and tree width, which are very important in Robertson and Seymour's work but had not been mentioned previously. These concepts are also applicable to making fast algorithms.

A **tree decomposition** of a graph G is a pair (T, X) where T is a tree indexing a family of subsets of $V(G)$. Each node $v_t \in V(T)$ has a corresponding subset X_t of the original graph $V(G)$ with the following properties:

1. For every edge $\{u, v\}$ of G, there exists a vertex t of T such that $u, v \in X_t$
2. For every pair y, z of vertices of T, if w is any vertex in the path between y and z in T, then $X_y \cap X_z \subseteq X_w$

The **width** of a decomposition is $max\{|X_i| - 1 | i \in V(T)\}$. The **tree width** of a graph G is the minimal k such that a tree decomposition of width k exists.

There are many graph problems that are believed to be not solvable in polynomial time based on the size of the input. Examples of such problems include:

- **Vertex Coloring:** Partitioning the vertices of a graph G into a minimum number of independent sets such that no edge joins two vertices in the same set.
- **Traveling Salesman Problem:** Finding the least cost cycle (where costs are assigned to edges) that visits every node exactly one (Hamiltonian Cycle).
- **Vertex Cover:** Find the smallest subset S of $V(G)$ such that every edge of G has at least one end in S.

However, when we restrict our input to graphs with a bound tree width, many of these hard problems (all of the examples) become solvable in polynomial time. See Algorithmic Implications of the Graph Minor Theorem for details on how [2].

7 Conclusions

Roberson and Seymour published a monumental body of work that contains many theoretically and algorithmically useful results, the vast majority of which were not covered here. However, hopefully this paper was able to provide enough necessary background for an excited reader to feel comfortable starting to explore their work. Anybody interested in graph theory will likely benefit from understanding more of their work, and there is certainly enough breadth to keep excited along with plenty of depth for anybody interested in details.

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