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Wronskians and Linear Independence

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Introduction

Determining the linear independence of a set of functions is an integral part of linear analysis as well as in finding solutions to differential equations. In *Wronskians and Linear Independence*, Bostan and Dumas [**2**] discuss the Wronskian matrix and the relation between its determinant and identifying linear dependence of functions. In particular, the Wronskian determinant for a family of functions will be identically zero if the functions are linearly dependent. Although this is a known result from linear analysis, Bostan and Dumas identify a common source of ambiguity as well as provide a new proof of the aforementioned fact.

Terminology

Definition. The Wronskian of a family of n functions that are $(n-1)$ times differentiable is defined as the determinant of the matrix of derivatives (Wronskian matrix).

$$det(W) = \begin{vmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & \vdots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Definition. The Vandermonde determinant $V(d_1, \ldots, d_n)$ is given by

$$V(d_1, \ldots, d_n) = \begin{vmatrix} 1 & \cdots & 1 \\ d_1 & \cdots & d_n \\ \vdots & \vdots & \vdots \\ d_1^{(n-1)} & \cdots & d_n^{(n-1)} \end{vmatrix} = \prod_{1 \leq i < j < n} (d_j - d_i)$$

Definition. The falling factorial $d(d-1)\cdots(d-k+1)$ is denoted by $(d)_k$.

Definition. A field $\mathbb{K}$ is an algebraic structure with addition, subtraction, multiplication and division defined and satisfying the group axioms.

Definition. The characteristic of a field is the fewest number of times the multiplicative identity must be summed to yield the additive identity. If the multiplicative identity never sums to the additive identity, the characteristic is said to be zero.

The Wronskian

It is not uncommon to see the term "Wronskian" used to refer both to the Wronskian matrix itself as well as its determinant; however, sensible nomenclature would have the term "Wronskian" refer to the determinant, and this is how we will use the term, as noted above. Furthermore, although it can be the case that linearly independent functions may have a zero Wronskian (the example given being the real functions x^2 and $x|x|$), if we restrict ourselves to analytic functions it does turn out that a zero Wronskian implies linear dependence. This is the hypothesis of theorem 1.

Theorem 1. *A finite family of linearly independent analytic functions has a nonzero Wronskian.*

A typical proof is that given by Anton [**1**, Chap. 5, pp. 246], wherein it is shown directly that a family of linearly dependent functions has an identically zero Wronskian. However, Bostan and Dumas elect to defer the proof of theorem 1 in favor of an extension from abstract algebra,

Theorem 2. *Le $\mathbb{K}$ be a field of characteristic zero. A finite family of formal power series in $\mathbb{K}[[x]]$, or rational functions in $\mathbb{K}(x)$, has a zero Wronskian only if it is linearly dependent.*

The following three lemmas are used in the proof of Theorem 2.

Lemma 1. *The Wronskian of the monomials $a_1x^{d_1}, \ldots, a_nx^{d_n}$ is equivalent to $C(d_1, \ldots, d_n)x^{d_1+\cdots+d_n-\binom{n}{2}} \prod_{i=1}^{n} a_i$.*

Although lemma 1 is never invoked directly, its result provides a nonzero matrix

$$D = \begin{vmatrix} 1 & \cdots & 1 \\ d_1 & \cdots & d_n \\ \vdots & \vdots & \vdots \\ (d_1)_{(n-1)} & \cdots & (d_n)_{(n-1)} \end{vmatrix}$$

which is useful in the proof of lemma 3.

Lemma 2. *Let $\mathbb{K}$ be a field and let $f_1, \ldots, f_n$ be a family of power series in $\mathbb{K}[[x]]$ which are linearly independent over $\mathbb{K}$. There exists an invertible $n \times n$ matrix A with entries in $\mathbb{K}$ such that the power series $g_1, \ldots, g_n$ defined by*

$$[g_1 \cdots g_n] = [f_1 \cdots f_n] \cdot A$$

are all nonzero and have mutually distinct orders. As a consequence, the following equality holds

$$W(g_1, \cdots, g_n) = W(f_1, \cdots, f_n) \cdot det(A)$$

Lemma two is proven by observing that since two series f_1 and f_2 are linearly independent, it is possible to find a linear combination such that $k_1f_1 + k_2f_2$ results in a series with strictly greater order than f_1. In the case of more than two functions the argument is repeated to show existence of the matrix A.

Lemma 3. *Let $\mathbb{K}$ be a field of characteristic zero. If the nonzero series $g_1, \ldots, g_n$ in $\mathbb{K}[[x]]$ have mutually distinct orders, then their Wronskian $W(g_1, \ldots, g_n)$ is nonzero.*

Lemma three is shown by noting that the Vandermonde determinant $V(d_1, \ldots, d_n)$ is nonzero if and only if the constituent d_i's are mutually distinct. The entries in the Wronskian $W(g_1, \ldots, g_n)$ can now be written as $w_{i,j} \times (1 + xr_{i,j})$ for some series $r_{i,j}$ in $\mathbb{K}[[x]]$, where $w_{i,j}$ is the (i,j)th entry in the original Wronskian. We can replace $w_{i,j}$ in the Wronskian with the new expression $w_{i,j} \times (1 + xr_{i,j})$ and the matrix D in lemma 1 can have its entries replaced by $(d_j)_{i-1} \times (1 + xr_{i,j})$, which yields a nonzero determinant.

From this, we can prove Theorem 2.

Proof of Theorem 2. Let $f_1, \ldots, f_n$ be linearly independent power series in $\mathbb{K}[[x]]$. Under this assumption, lemma 2 allows us to find series $g_k = \sum_{n=1}^{\infty} a_n z^n$ with mutually distinct orders such that the Wronskians of both the original functions $W(f_1, \ldots, f_n)$ and the power series $W(g_1, \ldots, g_n)$ are equal up to a nonzero factor. Following that, lemma 3 gives us $W(g_1, \ldots, g_n) \neq 0$, and since $W(g_1, \ldots, g_n) = K \cdot W(f_1, \cdots, f_n)$ we immediately obtain the result $W(f_1, \ldots, f_n) \neq 0$. In the second case of rational functions $f_1, \ldots, f_n$, the above result for power series extends to this case via a Laurent expansion $f_k(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. $W(f_1, \cdots, f_n) \neq 0$ in both cases. Thus, a finite collection of analytic or rational functions has a zero Wronskian only in the case that the functions are linearly dependendent.

Consequence

When finding a solution to a differential equation, it is not uncommon to generate multiple solutions. Since the Wronskian allows us to determine linear independence in a reasonably efficient way, it is of great utility in ascertaining whether extraneous solutions to ordinary differential equations are linearly dependent. Since the Wronskian is typically nonconstant, locating the zeros of the function that results from evaluating the Wronskian will identify areas near which the family of functions are linearly dependent; the case where $W \equiv 0$ naturally implies that the family of functions is linearly dependent everywhere.

Conclusion

A new and unique proof of the fact that a nonzero Wronskian implies linear independence was provided for families of analytic functions. Although this was not a new result, the proof given by Bostan and Dumas is noted as being particularly elegant and brief. In addition, due to its utility in the area of differential equations, the new proof provides an accessible route to understanding why the Wronskian is used to determine linear dependence rather than some other method.

References

[1] H. Anton and C. Rorres, *Elementary Linear Algebra*, 9th ed., Wiley, 2004.
[2] A. Bostan and P. Dumas, *Wronskians and Linear Independence*, American Mathematical Monthly, **117**(2010), 722-727.