Lie Groups

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1 Introduction

An important and often beautiful aspect of mathematics is the ability to combine distinct ideas in a way that either extends them or results in a brand new theory entirely. In the late 19th century, a couple of mathematicians by the names of Sophus Lie and Wilhelm Killing [1] did just that when they independently invented the idea of what we now call Lie algebras. Lie algebras were contrived as a means of studying differential equations using techniques akin to the study of algebraic equations, but they have evolved into a field of their research in their own right, in addition to revealing intimate ties with theoretical physics. Lie groups are endowed with a dual group/topological structure. As such, new discoveries about Lie groups produce results that extend naturally to other fields of math, making them a fascinating topic for current research.

2 Preliminaries

One of the more peculiar facets of mathematics is that we can possess a rigid definition of a structure (groups, vector spaces, manifolds, fields, etc.) but continue to make unexpected discoveries about them. Lie groups made their debut in the late 19th century as a means of studying continuous symmetry but have found innumerable applications in the field of theoretical physics, as an example. In order to appreciate the nature of Lie groups, we must first develop an intuition for the building blocks used to characterize them.

Algebraic Structures

Definition. *A* group *[2] is a set G together with a binary operation $\star$ defined on G satisfying the following:*

- $\star$ is associative. That is, $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$.
- G has an identity element e such that for all $a \in G$, $a \star e = e \star a = a$.
- for each $a \in G$ there is an inverse element $a^{-1} \in G$ such that $a \star a^{-1} = a^{-1} \star a = e$.

The group $(G, \star)$ is called *abelian*, or commutative if $a \star b = b \star a$ for all $a, b \in G$.

Definition. *We define an* algebra *to be a vector space V over a field F with a binary operation $\cdot : V \times V \to V$ if for $a, b \in F$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, the following are true.*

- $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{x} \cdot \mathbf{z}$

- $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$
- $(a\mathbf{x}) \cdot (b\mathbf{y}) = (ab)(\mathbf{x} \cdot \mathbf{y})$

These axioms tell us that the binary operation we are using, which is usually called multiplication, is *bilinear*. This, in turn, is another way of saying that $\cdot$ is linear in each argument, or that it satisfies the properties of linearity for each operand. Note that an algebra need not be associative.

Geometry

Definition. *A topological space is a set X, together with a collection of open subsets T with the following properties:*

- $\emptyset \in T$
- $X \in T$
- if $E_j \in T$, then $\bigcap_{j=1}^{N} E_j \in T$ A finite intersection of elements in T is in T.
- If $E_j \in T$, then $\bigcup_{j=1}^{\infty} E_j \in T$ An arbitrary union of elements in T is in T.

We can then discuss the topology of familiar spaces, say $\mathbb{R}$, where using the notation above we have $X = \mathbb{R}$ and T consists of all the open subsets of $\mathbb{R}$. We can see that these properties hold since the empty set is open, as is $\mathbb{R}$ itself and the other properties follow.

Definition. *A* manifold *[4] is a topological space that resembles Euclidian space locally. More specifically, a topological space $M \subseteq \mathbb{R}^n$ is a manifold if for every $x \in M$ there is an open set $U \subseteq M$ such that $x \in U$, and there is a continuous function $f : U \to \mathbb{R}^m$ (fixed m), with a continuous inverse.*

We call the number m, written above, the *dimension* of the manifold. For example, a sphere in $\mathbb{R}^3$ is a manifold of dimension 2 embedded into 3-space. Likewise, the unit circle in the plane is a 1-dimensional manifold. It is clear that we must have $n \geq m$ for this definition to make sense, but we shall note that a more surprising result says that n is also bounded above by roughly twice m.

Important Terminology

- Two topological spaces (X, T_1) and (Y, T_2) are *homeomorphic* if there is a continuous bijection $f : X \to Y$ with a continuous inverse.
- A *homomorphism* from a group $(G, \star)$ to a group $(H, \cdot)$ is a map $f : G \to H$ such that $f(x \star y) = f(x) \cdot f(y)$ for $x, y \in G$. Note difference in operations and where they are performed.
- Two groups are *isomorphic* if there is a bijective homomorphism between them.
- The *symmetric group* of a set Ω is the set of all bijections from Ω to itself with group operation composition, and is denoted by S_Ω.
- A *group action* of a group G on a set Ω is a map $f : G \times \Omega \to \Omega$ where $e \cdot c = c$ for $c \in \Omega$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for $a, b \in G$.
- An *automorphism* of a group G is an isomorphism from G to itself.

3 Characterization

The notions of groups, manifolds, etc. solicit further inquiry as subjects of their own, but it is also interesting to restrict our attention to structures that share properties of groups *and* manifolds. Until around 1930, they were called continuous groups, but by convention they are now called Lie groups, in honor of the man who first investigated them. We now formally define Lie groups and explore their connection to their corresponding Lie algebras.

Definition. *A Lie group over the field F is a group G that is also a differentiable manifold, for which the map*

$$\phi : G \times G \to G, (x, y) \mapsto xy$$

is also differentiable.

It turns out that the global structure of a Lie group is largely determined by the behavior at the identity element. For this reason, it is useful to study the tangent space T_eG at the identity element of G. When we impose an appropriate operation on this space we obtain a Lie algebra (often called the tangent algebra for this reason), which allows us to study the structure of Lie groups using the exponential map.

Lie Algebras

Definition. *A Lie algebra is a vector space V over a field F equipped with an anti-commutative binary operation $[\cdot,\cdot] : V \times V \to V$, that satisfies the Jacobi identity. That is, for $a, b \in F$,*

- $[ax + by, z] = a[x, z] + b[y, z]$
- $[x, x] = 0$ for all $x \in V$
- $[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$ for $x, y, z \in V$.

We denote define the Lie bracket to be that binary operation , where $[X, Y] = XY - YX$. We see that the Lie bracket evaluates to 0 if X and Y commute. It is easy to verify that the Lie bracket satisfies the required properties:

$$[ax + by, z] = axz + byz - (axz + byz) = a[x, z] + b[y, z]$$

$$[x, x] = xx - xx = 0$$

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = [x, yz - zy] + [y, xz - zx] + [z, xy - yx]$$

$$= (xyz - xzy) - (xyz - xzy) + (yxz - yzx) - (yxz - yzx) + (zxy - zyx) - (zxy - zyx) = 0$$

Connections

We will now prove a couple of basic but important theorems about uniqueness and and smoothness. The left translation of a topological group is denoted by $L(a) : x \mapsto ax$ below. If $f : M \to N$ is a C^∞ map beween C^∞ manifolds, we define the map

$$Tf(p) : T_pM \to T_{f(p)}N$$

which is a linear transformation.

Theorem. *Let G be a Lie group and $\mathfrak{g}$ be it's tangent algebra. Take $\xi \in T_eG$ and $\eta \in \mathfrak{g}$. Then there is exactly one analytic homomorphism $\gamma : \mathbb{R} \to G$ such that $f'(0) = \xi$. This says that γ is the maximal integral curve of η through the identity of G or $\gamma(t) = \eta(\gamma(t))$.*

Proof. Let $I = (-\epsilon, \epsilon)$ and define the maps

$$u : s \mapsto \alpha(t+s) \quad \text{and} \quad v : s \mapsto \alpha(t)\alpha(s)$$

where $\alpha : \mathbb{R} \to G$ is the integral curve defined on subintervals of I containing 0, that maps 0 to e. Defining $\rho : x \mapsto t + x$ allows us to rewrite $u(s) = (\alpha \circ \rho(t))(s)$ and $v(s) = \left(L(\alpha(t)) \circ \alpha\right)(s)$

$$\begin{aligned}
u'(s) &= \frac{d}{du} T(\alpha \circ \rho(t))(s) \\
&= T\alpha(\rho(t)(s)) \circ \frac{d}{du} T\rho(t))(s) \\
&= \frac{d}{du} T\alpha(t+s) \\
&= \alpha'(t+s) \\
&= \eta(\alpha(t+s)) \\
&= \eta(u(s))
\end{aligned}$$

which we obtain from the chain rule and from the way which we defined our auxiliary functions. We employ a similar process for $v(s)$:

$$\begin{aligned}
v'(s) &= \frac{d}{du} T(L(\alpha(t)) \circ \alpha)(s) \\
&= \frac{d}{du} (L(\alpha(t)) \circ \alpha))(s) \\
&= TL(\alpha(t))(\alpha(s) \circ \frac{d}{du} T\alpha(s) \\
&= TL(\alpha(t))(\alpha(s))\alpha'(s) \\
&= TL(\alpha(t))(\alpha(s))\eta(\alpha(s)) \\
&= TL(\alpha(t)))\eta(\alpha(s)) \\
&= (\eta \circ L(\alpha(t)))(\alpha(s)) \\
&= \eta(\alpha(t)\alpha(s) \\
&= \eta(v(s))
\end{aligned}$$

Observe that $u(s)$ and $v(s)$ are solutions to the differential equation $y'(s) = \eta(y(s))$ satisfying conditions $u(0) = v(0) = \alpha(t)$. We utilize an unstated theorem to ascertain that there is a neighborhood N of 0 on which α is a local analytic homomorphism. This gives us the first part of our desired result. For the uniqueness, we let $t = t_1 + t_2$ where $t_1, t_2 \in N$ and set $f = \alpha$. Then since f is a homomorphism we have $f(t_1 + t_2) = f(t_1)f(t_2)$. Using our definitions and previous result, we compute

$$
\begin{aligned}
\eta(f(t)) &= \eta(f(t_1 + t_2)) \\
&= \eta(L(f(t_1)f(t_2)) \\
&= TL(f(t_1))(f(t_2))f'(t_2) \\
&= TL(f(t_1))(f(t_2)) \cdot \frac{d}{du}Tf(t_2) \\
&= T(L(f(t_1)) \circ \frac{d}{du}f)(t_2) \\
&= T(f \circ \frac{d}{du}\rho(t_1))(t_2) \\
&= (Tf)(t_1 + t_2) \circ \frac{d}{du}(T\rho(t_1))(t_2) \\
&= \frac{d}{du}(Tf)(t) = f'(t)
\end{aligned}
$$

Because we can write any $t \in \mathbb{R}$ in the form $\sum_{j=1}^{n} t_j$ we obtain our result by doing induction on n and remembering that f is a homomorphism.

□

Since the exponential map bridges a Lie group and its algebra, we would like to know more about its properties. Here we state such a result.

Definition. *Let G be a Lie group and $\Phi \in \mathfrak{g}$ be the vector field corresponding to $\Omega \in T_eG$. Then let ϕ denote the homomorphism from our last result. We can now generalize our definition of the exponential to be given by*

$$\exp : \mathfrak{g} \to G, \quad \Phi \to \phi(1).$$

Theorem. *Consider a Lie group G with Lie algebra $\mathfrak{g}$. Then the exponential map $\exp : \mathfrak{g} \to G$ is analytic and $T(\exp)(0) : T(\mathfrak{g}, 0) \to T_eG$ is invertible.*

Proof. Our previous result tells us that there is a neighborhood $U \subset T_eG$, a neighborhood $W \subset \mathbb{R}$ and a map $h : U \times W \to G$ which is analytic, and for each $t \in W$ and $s \in U$, $h(t, s) = \phi(t)$. Now pick a number $\lambda \in W$ such that $0 < \lambda < 1$ and consider the set $S = \lambda U \subset U$. We can see that the map $S \to G : \Omega \to h(\lambda, \lambda^{-1}\Omega)$ is well-defined and analytic since $\lambda \in W$ and $\lambda^{-1}\Omega \in \lambda^{-1}S = U$. At this point we invoke a map which we will take to be an isomorphism, $\varphi : \mathfrak{g} \to T_eG$, which is defined by $\Phi \to \Omega$. Since φ is linear and analytic, we get analyticity of $\exp \circ \varphi^{-1}$ at $0 \in T_eG$.

Now we need to show that $\exp \circ \phi^{-1}$ is analytic for all $\Omega \in T_eG$. For some neighborhood E of Ω, there is some $\delta > 0$ such that $\delta^{-1}E \subset W$. But exp is analytic on W so $\exp(\delta^{-1}\Omega)^{\delta} = \exp \Omega$ is analytic on E. We then observe that $T_eG \cong \mathfrak{g}$ by φ^{-1} so $(\exp \circ \varphi^{-1}) \circ \varphi$ is analytic on $\mathfrak{g}$.

To get the invertibility result we use the fact that the tangent space of the finite dimensional vector space $\mathfrak{g}$ at the point 0 is isomorphic to the vector space of all the directional derivatives evaluated at 0. This is to say that $D_\Phi(0) \cong \mathfrak{g}$ where D_x is the vector space of directional derivatives and x is one of those vectors. We'll let γ be the isomorphism between these spaces. We now define $f_\Phi(t) = \exp(t\Phi)$ and let Q be a function that is analytic at $e \in G$. Then we can write

$$\begin{aligned}T(\exp(0))(D_\Phi(0)(Q) &= D_\Phi(0)(Q \circ \exp)\\ &= \lim_{t\to 0}\frac{1}{t}(Q\circ f_\Phi(t) - Q\circ f_\Phi(0))\\ &= \frac{d}{dt}(Q(e\cdot f_\Phi(0)) = (\Phi\circ Q)(e) = \Omega(Q)\end{aligned}$$

which requires some justification which we will leave to the curious reader. We can now write $T(\exp)(0)\cdot D_\Phi(0) = \Omega$. But since we have isomorphisms φ and γ, we also know $\varphi(\Phi) = \Omega$ and $\gamma(D_\Phi)(0) = \Phi$ which yields invertibility of $T(\exp)(0) = \varphi\circ\gamma$.

□

Basic Examples

Many structures the reader is likely familiar with can be classified as Lie groups. A typical example of a Lie group is the boundary of the unit disk in the complex plane. It's identity element is $z = 1$ and tangent space, the line $z = 1 + it$ with addition is the corresponding Lie algebra. In this case, it is obvious that the exponential map, $\xi \mapsto e^{i\xi}$, sends elements of the line to the circle. Because More generally, we denote the group of invertible $n\times n$ matrices with matrix multiplication by $GL(n, F)$, [3] also called the general linear group. The corresponding general linear algebra is written as $\mathfrak{gl}(V)$. Here, F is either of the fields $\mathbb{R}$ or $\mathbb{C}$ and V is a vector space. Also important are the special linear group $SL(n, F) = \{A \in GL(n, F) \,|\, \det A = 1\}$, which is a *subgroup* of the general linear group, and the special orthogonal group, $O(n, F) = \{A \in GL(n, F) \,|\, AA^T = A^TA = I\}$. It is easy to check that both of these are subgroups of $GL(n, F)$.

4 Representation

A *representation* [5] of a Lie algebra $\mathfrak{g}$ is a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$. The map φ sends elements $x \in \mathfrak{g}$ to elements $\varphi(x) \in \mathfrak{gl}(V)$. From the definition of a homomorphism, we see that for $X, Y \in \mathfrak{g}$

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)] = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X),$$

and

$$\varphi(\alpha X + \beta Y) = \alpha\varphi(X) + \beta\varphi(Y)$$

from linearity of V. This is useful as any Lie group that we can represent in this fashion can be understood in terms of matrices.

Root Systems

In linear algebra, we represent a vector space V with a basis $B = \{e_1, e_2, \cdots, e_n\}$ such $\text{span}\{e_1, e_2, \cdots, e_n\} = V$. This basis could be composed of vectors, matrices, or any object for which addition and multiplication can be defined. The generalization of this idea is a *root system*, where the roots are analogous to the basis elements and represent the independent "directions" in the structure.

Let $(\cdot,\cdot)$ be the usual inner product on a vector space V, and $\sigma_r(s)$ be the reflection of s through the hyperplane perpendicular to r. A root system is a set U defined on V where

i. $\text{span}\{e_1, e_2, \cdots, e_n\} = V$, for $e_j \in U$

ii. for $x \in U$, $\alpha x \in U$ iff $\alpha = 1$ or -1

iii. for $r, s \in U$, $\sigma_r(s) = s - 2r(r,s)/(s,s) \in U$

Furthermore, a root system R is called *irreducible* if there is no subset $S \subset R$ that spans V and where $(\alpha, \beta) = 0$ for all $\alpha \in S$ and $\beta \in R$ but $\beta \notin S$. We can extent the utility of a root system to the directed graph representation of a Lie group, a Dynkin diagram. These diagrams depict roots as nodes in a graph with the edges are determined by orthogonality relations between the roots. This is a more difficult endeavor which we will return to at another time.

5 Conclusion

Lie groups have proven to be interesting structures in many regards. The fact that they can be studied almost entirely by their local behavior allows us to investigate them via algebraic techniques, as well as those from topology. But they also lie in a smaller class of objects due to the requirements we place on them, giving rise to more specific and interesting properties. At this point it is clear that Lie groups have deep aesthetic beauty, but they also have applications in physics. Since they were originally pondered with eigenvalue problems in mind, they are often used to reveal and understand symmetries in differential equations. For this reason they have piqued the interest of particle physicists, yielding new insights into particle symmetries in the standard model. In our current state, there is important work going on in classifying and representing Lie groups, a bold undertaking which will almost certainly lead us to new and interesting mathematical insights.

References

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