The No Retraction Theorem and a Generalization

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1 Introduction

The No-Retraction Theorem is a result in point-set topology concerning functions which map topological spaces to their boundaries. Roughly, the Theorem states that there does not exist a continuous function from the unit disk to its boundary which, restricted to the boundary, is the identity. It is equivalent to the Brouwer Fixed-Point Theorem, which states that any continuous automorphism of the unit disk has at least one fixed point. The possibility of generalizing both theorems to topological spaces more interesting than those homeomorphic to the unit disk is enticing. This paper presents one such generalization of the No-Retraction Theorem, yielding a result which applies to a broad class of topological spaces.
In particular, the No-Retraction Theorem can be extended to those topological spaces homeomorphic to what are called 2-dimensional simplicial complexes. Simplicial complexes should be thought of as finite collections of triangles embedded in Euclidean space, which intersect only at edges and vertices. We must generalize the concept of boundary to this class of topological spaces, and then demonstrate that a close analogue of the No-Retraction Theorem holds for these definitions.

2 Definitions

Some basic definitions, and the No-Retraction Theorem, formally.

Definition 1 (Topological Space)**.** For our purposes, the term topological space simply refers to some subset of Euclidean space, $\mathbb{R}^n$. Furthermore, the topological space has a relative definition of open and closed sets. If X is a topological space, then a set $A \subset X$ is open in X if A is the intersection of X with an open subset of $\mathbb{R}^n$. Similarly, A is closed if it is the intersection of X with a closed subset of $\mathbb{R}^n$.

Definition 2 (Retraction)**.** If S is a subset of $\mathbb{R}^n$, and $B \subset S$, then a retraction $r : S \to A$ is a continuous mapping such that $r(b) = b$ for all $b \in B$.

Definition 3 (Homeomorphism)**.** A homeomorphism from one topological space to another is a *bi*continuous (and thus invertible) mapping. Homeomorphisms preserve a wide variety of topological properties. If there exists a homeomorphism between two spaces, they are said to be **homeomorphic** to each other.

Finally, the result to be generalized:

Theorem 1 (No-Retraction Theorem (NRT) in $\mathbb{R}^2$)**.** *There is no retraction from the closed unit disk,* $\mathbb{D} = \{\mathbf{x} : |\mathbf{x}| \leq 1\}$ *to its boundary the unit circle,* $C = \{\mathbf{x} : |\mathbf{x}| = 1\}$.

Proof. We proceed by contradiction. Suppose $f : \mathbb{D} \to S^1$ is a retraction from the unit disk to its boundary. Let α, ω be points on the unit circle: $\alpha, \omega \in S^1$. Note that $S^1 \setminus \{\alpha, \omega\}$ is disconnected, as it is the union of two disjoint relative open sets. Thus, it suffices to show that it is the image under f of a connected subset of $\mathbb{D}$. Let A denote the preimage of $\{\alpha\}$ under f. Now $A \subset \mathbb{D}$ may be disconnected, but it certainly intersects S^1 at only one point, that is, at α, since f, restricted to S^1, is the identity. Similarly, $W = f^{-1}(\{\omega\})$ only intersects S^1 at ω. The two sets A and W may intersect themselves, so excising them may disconnect $\mathbb{D}$. However, because they each intersect S^1 at only one point, excising them leaves intact some subset of $\mathbb{D}$ whose closure includes the entirety of S^1. Let this set be denoted by E. Now E is connected, and it is open, since A and W are closed. Thus, it is path connected.

Let α_0 and α_1 be endpoints of a small arc on S^1 centered at α. They are both contained in the closure of the path connected set E discussed above, so there exists a continuous path γ connecting them which does not intersect A or W. But the union of this path with $S^1 \setminus \{a, b\}$ is connected—this can be seen by simply considering the union of γ with some path from α_0 or α_1 to an arbitrary point on $S^1 \setminus \{a, b\}$. Finally, we note that $f(S^1 \setminus \{\omega, \alpha\} \cup \gamma) = S^1 \setminus \{\alpha, \omega\}$, because γ avoided both A and W. Thus, we have a contradiction. [1] □

By composition of mappings, the NRT applies to all spaces which are homeomorphic to the unit disk.

3 Simplicial Complexes

Above, we briefly sketched the features of 2-dimensional simplicial complexes. First, we define what we mean by "triangle".

Definition 4 (2-Dimensional Simplex)**.** Let S denote the unit simplex in $\mathbb{R}^2$, given by $S = \{\mathbf{x} : x_1, x_2 \geq 0; ||\mathbf{x}||_1 \leq 1\}$. A topological space $T \subset \mathbb{R}^n$ is a 2-dimensional simplex if T is an affine linear transformation of S. Note that 2-dimensional simplices have 3 vertices and 3 edges, which are respectively the images of the vertices and edges of the unit simplex, and are "flat" (subsets of 2-dimensional vector spaces).

We can now make sense of the finite union of triangles:

Definition 5 (2-Dimensional Simplicial Complex)**.** A 2-dimensional simplicial complex K is a collection of finitely many 2-dimensional simplices T_i, with the nonempty intersections $T_i \bigcap T_j, i \neq j$ consisting only of the shared edges and vertices of T_i and T_j. The underlying space of K, denoted $|K|$, is the point-set union of the triangles of K.

Definition 6 (Subdivision of a Simplicial Complex)**.** A subdivision of a 2-dimensional simplicial complex K is another complex obtained by decomposing the simplices of K into smaller triangles, while maintaing all the intersections of K.

It should be noted at this point that 2-dimensional simplicial complexes are the higher-dimensional variants of graphs. A (simplicial) graph consists of the union of line segments which intersect only at their shared vertices.

Because 2-dimensional simplicial complexes can be embedded in more than 2 dimensions, it follows geometrically and intuitively that edges can belong to three or more simplices. In the case of simplicial

[1] I am indebted to Sourav Chakraborty for the idea for this proof.

complexes that we can draw on paper, restricted to 2 dimensions, it is clear what we should consider the boundary: those edges which form the topological boundary of the underlying space of the complex. These are those edges which are contained in only one triangle. However, we can imagine cases in more dimensions where this simple definition is insufficient. One such example is a topological space consisting of three disks (two of them stretched outward to form hemispheres) glued together at their common boundary, and with two tubes connecting the middle disk to its two neighbors:

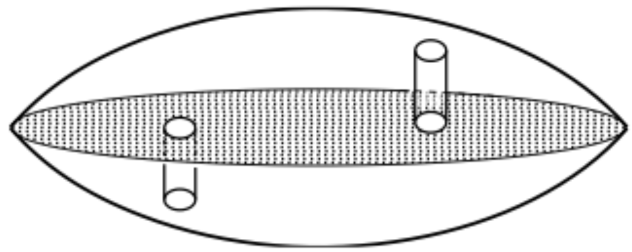

Figure 1: Topological space homeomorphic to a simplicial complex

Without demonstrating it here, this shape is homeomorphic to the underlying topological space of a 2-dimensional simplicial complex. But it does not have a boundary in the sense of having any edges which are members of only one simplex. The notion of boundary requires a generalization to include spaces of this sort:

Definition 7 (Boundary of a 2-Dimensional Simplicial Complex)**.** Let K be a 2-dimensional simplicial complex. Then the boundary of K, denoted $\operatorname{Bd} K$, is the set of all edges of K which are contained in an odd number of triangles.

Because homeomorphisms preserve boundaries, we can speak of the boundary of any topological space X homeomorphic to the underlying space of a 2-dimensional simplicial complexes. Intuitively, this should be those points in x which map homeomorphically to an edge in the complex contained in an odd number of triangles. We can formalize this intuition by defining T_i to be the graph obtained by joining i copies of a unit line segment together at a common vertex. We denote this vertex by $\{*\}$. Then $T_i \times \mathbb{R}$ locally resembles an edge shared by i triangles:

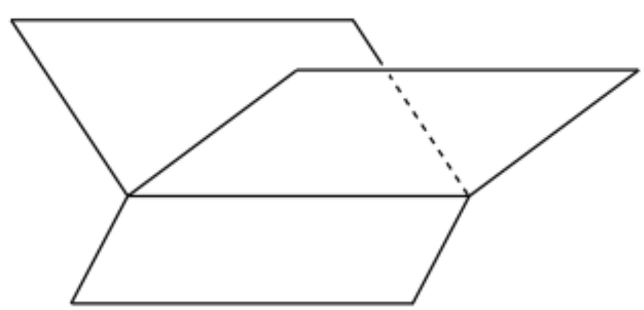

Figure 2: Example of a generalized boundary

Definition 8. Let X be a topological space, and let C_i be the set $\{x \in X : f(N(x)) = T_i \times \mathbb{R}, f(x) = \{*\} \times \mathbb{R}\}$, where $N(x)$ is a neighborhood of x, and f is a homeomorphism. Then $\partial X = \bigcup_{i \text{ odd}} C_i$. That is, the boundary of X is the union of those points at which X is locally homeomorphic to the boundary of a 2-dimensional simplicial complex.

We note here one more important fact about the generalized definition of boundary.

Proposition 1. If K is a 2-dimensional simplicial complex, then $\partial|K| = |\operatorname{Bd} K|$.

4 The Generalized No-Retraction Theorem

Armed with the appropriate generalization of a boundary on 2-dimensional simplicial complexes, we are equipped to provide the corresponding generalization of the No-Retraction Theorem. First, however, we require a simple lemma from graph theory.

Lemma 1. *Let G be a graph with no loops, that is, no edges whose vertices are identical with each other. Then the number of vertices of G contained in an odd number of edges is even.*

Proof. Begin with the graph G which has no loops. Remove edges one by one. For each removal, there are a number of cases. If the edge E is isolated, sharing its vertices with no other edges, then its removal reduces the number N of odd-edged vertices by 2. Suppose E has one vertex of degree one, and one vertex shared with p other edges. Then N is reduced by 2 if p is even, and unchanged if p is odd. Finally, suppose E shares its vertices with p and q other edges, respectively. If p and q have different parity, N is unchanged. If they have the same parity, then N is reduced or increased by 2. In every case, the parity of N is unchanged. By removing edges, we can reduce any non-looping graph to the base case of a single isolated edge, which clearly has $N = 2$. Thus, N of G was even. □

This lemma will be crucial to the proof of the generalized No-Retraction Theorem, which we state here.

Theorem 2 (Generalized No-Retraction Theorem)**.** *Let X be a topological space homeomorphic to the underlying space of a 2-dimensional simplicial complex. Then there exists no retraction $r : X \to \partial X$.*

Proof. Let X be homeomorphic to $|K|$, for some 2-dimensional simplicial complex K. By Proposition 1 and composition of continuous functions with homeomorphisms, it suffices to show that there is no retraction $r : |K| \to |\operatorname{Bd} K|$. Suppose that there is such a retraction. We proceed by contradiction.

Step 1

Let Δ denote an equilateral triangle with edges of length 1, which is disjoint from K. Call one of its edges η, and let ω be the vertex opposite η. Let σ be an edge in the boundary of K. Then σ belongs to an odd number of triangles of K.

Define a continuous map $h : |K| \to \Delta$ by mapping σ surjective affine linearly onto η. Map every vertex v of $|K| - \sigma$ onto ω, and extend affine linearly over all the simplices of K. That is, $h(\omega) = \eta$, $h(v) = \omega$, and all of the edges of K are mapped into $\partial\Delta$. Indeed, if a simplex T of K does not intersect with σ, then all its vertices are mapped to ω, and by linear extension, $h(T) = \omega$. If T contains one vertex of σ, then it is mapped to one of the edges of Δ. And if T contains σ, then its vertices map onto the vertices of Δ, and $h(T) = \Delta$. Thus, the edges of T map onto the edges of Δ. So $h(E) \in \partial\Delta$ for all edges E of K. As a corollary, note that h takes $|\operatorname{Bd} K|$ to $\partial\Delta$. Furthermore, the only edge of $\operatorname{Bd} K$ that maps surjectively onto

η is σ, since all the other edges have a vertex mapping to ω.

We define $r' = h \circ r$, where r is the retraction we supposed to exist above. Now $r' : |K| \to \partial\Delta$, because the image of r is a subset of $|\operatorname{Bd} K|$. By the properties of h above, the only points in $|\operatorname{Bd} K|$ that r' maps to η are those in σ. The preimage of η may have other points in $|K|$, because r is of course not injective. But restricting to the boundary, $(r'|_{|\operatorname{Bd} K|})^{-1}(\eta) = \sigma$. Clearly r' is continuous, which, combined with the compactness of $|K|$, means that it is uniformly continuous. Let δ be such that if $|x_1 - x_2| < \delta$, then $|r'(x_1) - r'(x_2)| < 1/8$. In particular, the images of triangles with diameter less than δ will have diameter less than $1/8$. Let L be a subdivision of K all of whose simplices have diameter less than δ.

Define a map $f : |L| \to \Delta$ by taking $f(v) = r'(v)$ for all vertices v of L, and extending affine linearly over the simplices of L. As we saw, r' maps into the boundary of Δ, so $f(v) \in \partial\Delta$. By Proposition 1, $|\operatorname{Bd} L| = |\operatorname{Bd} K|$, so $r(|\operatorname{Bd} L|)$ is the identity. The properties of h guarantee that f maps the edges of $\operatorname{Bd} L$ to $\partial\Delta$. However, f does not necessarily map all the edges and simplices of L into $\partial\Delta$. What we can say is that, since all the vertices of L do map into $\partial\Delta$, and the diameter of the image of each triangle, $f(T)$, is less than $1/8$, that the image of L under f is a subset of $\partial\Delta$ along with three equilateral triangles of diameter $1/8$ at the points of Δ.

Recall the edge η? Pick $y \in \eta$ (but not a vertex of η) such that y is not the image of a vertex of L, and y is at most a distance of $1/8$ from the midpoint of η. This is possible because L has finitely many vertices. We have chosen y to guarantee that its preimage is a subset of triangles which map into η. Now, we saw earlier that $(r'|_{|\operatorname{Bd} K|})^{-1}(\eta) = \sigma$, so the only points in $|\operatorname{Bd} L|$ which map to η are σ. The function r' was defined to map σ affine linearly onto η, that is, homeomorphically. Furthermore, we defined f to be equal to r' at the vertices of L, and extend linearly elsewhere. If we consider the map r' from σ to η, now subdivided into a number of vertices v, the images of these vertices are simply distributed, with their order preserved, along η. Extending affine linearly to form $f(\sigma)$ over all of η gives f homeomorphic on σ itself. In particular, homeomorphisms are injective, so the preimage of y restricted to σ is a single point. But $(f|_\sigma)^{-1}(y) = (f|_{|\operatorname{Bd} L|})^{-1}(y)$, so the latter consists of just one point.

Step 2

Whereas the preimage of y restricted to the boundary of L is just a single point, this is likely not the case for the preimage of y considered everywhere. The map f takes the simplices of L into η affine linearly, and we have chosen y to lie on the image of none of the vertices of L. Thus, it lies in the image of the interiors and edges of some of the simplices of L. Because f is affine linear, the preimage of y forms a graph G. The edges of G are the nonempty intersections of $f^{-1}(y)$ with the simplices of L, and its vertices are the nonempty intersections of $f^{-1}(y)$ with the edges of L. So the triangles in L containing edges of G are those taken by f into η which have y in the relative interior of their images. Because f is affine linear, no such triangle contains more than one edge of G. If it did, then we would have two disjoint line segments in T mapping affine linearly to a single point.

Let v be a vertex of G. We have $v \in f^{-1}(y)$, and that v is contained in the relative interior of an edge μ of L. Let T be a triangle of L that has μ as one of its edges. So $f(\mu)$ is in η and contains y in its relative interior. Then y is in the relative interior of the image under f of T, and T contains an edge of G. Thus, every triangle containing μ contains an edge of G. So the number of edges of G containing v is equal to the number of triangles of L containing μ. In particular, if μ is in the boundary of L, then v is contained in an odd number of edges of G. Since f is affine linear, all the edges of G are straight lines, and thus G contains no loops. So by Lemma 1, the number of vertices of G contained in the boundary of L is even. This contradicts the earlier result that this number was one. $\square$

5 Further Considerations

The above result is a generalization of the No-Retraction theorem, but it is still particular to 2 dimensions. A similar proof of higher-dimensional cases should be possible. In the 3-dimensional case, simply extrapolating each object of the proof to the 3-dimensional analogue gives us a retraction of a regular tetrahedron. And considering the preimage of one point on a face of this tetrahedron, it is a graph of the same type. That is to say, the essential ideas, and in particular the critical graph-theoretical lemma, are still applicable in higher dimensions.

6 References

1. E. D. Bloch, A Simple Proof of a Generalized No Retraction Theorem, *American Mathematical Monthly.* 116 (2009), 342–350.
2. Sourav Chakraborty, Point Set Topological Proof of “no-retraction” Theorem for 2 and 3 dimensional cases. www.cs.uchicago.edu/∼sourav/papers/topology.pdf