On Infinitely Nested Radicals

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1. Introduction

In "On Infinitely Nested Radicals" by Seth Zimmerman and Chungwu Ho, which appeared in the Feb. 2008 issue of Mathematics Magazine [1], the questions of convergence, density, and correspondence of rational numbers that can be written as infinitely nested radicals are explored. One example of a nested radical is the Golden Ratio, ϕ, which can be written as:

(1.1) $$\phi = \sqrt{1+\sqrt{1+\sqrt{1+}}}\ldots$$

Many questions arise from the study of nested radicals, such as:

1. What can we say about the convergence of them?

2. What types of numbers (i.e. rationals, integers, transcendentals etc.) can we write as nested radicals?

3. For a number k, how many different sequences will converge to k?

4. If we take all possible limits of sequences, what can we say about their density?

2. An Example

We begin by looking at a relatively simple example of a nested radical:

(2.1) $$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\ldots$$

We need to have a formal definition of what a nested radical actually is, so let us define the sequence as follows:

(2.2) $$x_1 = \sqrt{2}, x_{n+1} = \sqrt{2+x_n} \quad (n \geq 1)$$

Clearly when we extend Equation 2.1 out infinitely it is equal to $\lim_{n \to \infty} x_n$. Furthermore when we look at the sequence $\{x_n\}$ we can use the Half-Angle identity to see that $x_n = 2 \cdot cos(\pi/(2^{n+1}))$. Thus the sequence $\{x_n\}$is bounded and monotonically increasing, and therefore converges. We can also see that as $n \to \infty$, $2^{n+1} \to \infty$, so $\pi/(2^{n+1}) \to 0$ and since $cos(0) = 1, x_n \to 2.$ We will extend this idea in the next section by taking an arbitrary value a instead of 2.

3. Radicals of the form $\sqrt{a+\sqrt{a+\sqrt{a\ldots}}}$

As before we will define $r_1(a) = \sqrt{a}$ and $r_{n+1}(a) = \sqrt{a + r_n(a)}$ for some rational number a and begin by looking at the limit of the sequence of r_n's as n approaches infinity. We can very simply show that the sequence $\{r_n\}$ is increasing and furthermore inductively show it is bounded. Now, looking at $r(a) = \lim_{n \to \infty} r_n$, we use the equation $r_{n+1} = \sqrt{a + r_n}$. Squaring both sides and moving all terms to one side of the equation leaves us with the quadratic $r^2 - r - a = 0$ which has roots $r = (1 \pm \sqrt{1+4a})/2$. But we also know that $a > 0$ so $\sqrt{1+4a} > 1$, and since $r > 0$ we can remove the possibility of subtraction and conclude that:

(3.1) $$r(a) = (1/2)\cdot(1 + \sqrt{1+4a})$$

This shows that the correspondence between a and r is one-to-one. Furthermore since a is rational and r must be the roots of a quadratic with rational coefficients, r cannot be transcendental.

THEOREM 1: *For each rational number* $h > 1, h(h-1)$ *is the unique rational number a such that* $r(a) = h$.

Proof: Fix h and let $a = h(h-1)$. Then we can substitute h in for a in Equation 3.1, and after simple algebraic manipulation we obtain the desired result. Uniqueness comes from the fact that Equation 3.1 is a one-to-one correspondence.

4. Radicals of the form $\sqrt{a + b\sqrt{a + b\sqrt{a + \ldots}}}$

We will use the same definitions as in Section 3, $s_1 = \sqrt{a}, s_{n+1} = \sqrt{a + b \cdot s_n}$ and let $r_n = (1/b)\cdot s_n$, where r_n is from Section 3. We will also define $\lim_{n \to \infty} \{s_n(a,b)\} = s(a,b)$. 3. Inductively we can show that:

(4.1) $$r_{n+1} = \sqrt{(a/b^2) + r_n}$$

Thus we get $s_n = b \cdot r_n(a/b^2)$. Using our results form the previous section we show that $s = (1/2)\cdot(b + \sqrt{b^2 + 4a})$, and furthermore we can apply $s_{n+1} = \sqrt{a + b \cdot s_n}$ to show that s must be a root of the equation $x^2 - bx - a = 0$ and therefore again a transcendental will not be the limit of this type of nested radicals. Clearly, if we choose s, then there are infinitely many possible choices of a and b that will produce an equivalent nested radicals relation.

5. Radicals of the form $\sqrt{a-b\sqrt{a-b\sqrt{a-...}}}$

Things begin to get considerably trickier when we start to deal with these types of nested radicals. For starters we will set $b=1$ and define the sequence $\{u_n\}$ similar to before, with $u_{n+1}=\sqrt{u-u_n}$. Thus if $u_n \le \sqrt{a}$ then u_{n+1} also is. To avoid imaginary numbers we restrict $a>1$. Inductively we can show the sequence of even terms is increasing and the sequence of odd terms is decreasing. They are both also bounded between 0 and $\sqrt{a}$ and thus converge. We can furthermore define a function $u_{n+1}+u_n=f(x)=\sqrt{a-x}+x$ on the interval $[0,\sqrt{a}]$, differentiate, and see that the minimum for this function must lie at an endpoint of the interval. We can use this to show the minimum is obtained at the left endpoint $x=0$ and that over the entire interval $u_{n+1}+u_n \ge \sqrt{a}$. We will use this to show that the two sequences of odd and even terms converge to the same limit:

$$\begin{aligned}|u_{n+1}+u_n| &= \left|u_{n+1}^2-u_n^2\right|/(u_{n+1}+u_n)\\ &=|a-u_n-a+u_{n-1}|/(u_{n+1}+u_n)\\ &\le |u_n-u_{n-1}|/\sqrt{a}\end{aligned}$$

Since $a>1$, both even and odd subsequences thus converge to the same limit and therefore $\{u_n\}$ converges. From our earlier equations we can see that $x^2+x-a=0$ must be satisfied by u, and thus

(5.1) $u(a)=(1/2)\cdot(-1+\sqrt{1+4a^2})$.

When we take $b>1$, we define $v_{n+1}=\sqrt{1-b\cdot v_n(a,b)}$ and thus, $v_n(a,b)=b\cdot u_n(a/b^2)$. The sequence $\{u_n\}$ will only be defined when $a_n \ge b_n^2$. We will use this result to prove Theorem 2.

THEOREM 2: *Let h, a, and b be positive numbers. Then* $h=\lim_{n\to\infty} v_n(a,b)$ *if and only if:*

1. $0<b<\phi\cdot h$, *and*
2. $a=h(h+b)$

Proof: 2. Since $v_n(a,b)=b\cdot u_n(a/b^2)$, $\lim_{n\to\infty} v_n(a,b)$ exists if and only if $u=\lim_{n\to\infty} u_n(a/b^2)$ and $v=b\cdot u$. From Equation 5.1 we get that $v(a,b)=b\cdot(1/2)\cdot(-1+\sqrt{1+4(a/b^2)})$. Thus $v(a,b)=(1/2)\cdot(-b+\sqrt{b^2+4a})$.

Algebraic manipulation shows this is true if and only if $a=h(h+b)$. Thus condition 2 is proven.

1. For u_n to exist entirely in the reals we earlier specified that $a \ge b^2$. However we have also established $a=h(h+b)$, leading to the inequality $h(h+b)>b^2$ (the inequality is strict because equality leads to an alternating series of 0's and b's). Thus $b^2-hb-h^2<0$. When we fix h, we notice that the quadratic opens upward and thus the inequality is only satisfied when b lies between the two roots: $(1-\sqrt{5})h/2<b<(1+\sqrt{5})h/2$. But $b>0$ so $0<b<\phi\cdot h$ and condition 1 is reached.

COROLIARY: *For each positive integer k, there exist integers a and b such that* $k=v(a,b)$. *Furthermore, since a and b have to satisfy the conditions of Theorem 2, there are finitely many of them.*

6. Alternating Series Radicals

We now turn to the most complex type of nested radicals that we will encounter in this paper, those of the form: $\sqrt{a+b\sqrt{a-b\sqrt{a+b\sqrt{a-\ldots}}}}$ or their alternate, $\sqrt{a-b\sqrt{a+b\sqrt{a-b\sqrt{a+\ldots}}}}$.

We will start by recursively defining a pair of sequences $x_1=\sqrt{a+b\sqrt{a}}$ and $y_1=\sqrt{a-b\sqrt{a}}$ with their recursive definitions $x_{n+1}=\sqrt{a+b\cdot y_n}$ and $y_{n+1}=\sqrt{a-b\cdot x_n}$. First we need to see that these converge. The x's have no problems converging if the y's are positive, but the y's bring up a more complicated case.

THEOREM 3: *Let a and b be positive real numbers such that* $a>b\sqrt{a+b\sqrt{a}}$. *Then for each positive integer n,* $0<y_n\leq\sqrt{a}\leq x_n\leq\sqrt{a+b\sqrt{a}}$.

Proof by Induction: Let $n=1$. Then $a>b\sqrt{a+b\sqrt{a}}>b\sqrt{a}$. So since $y_1=\sqrt{a-b\sqrt{a}}$, $y_1>0$. Furthermore, $y_1=\sqrt{a-b\sqrt{a}}<\sqrt{a-0}<\sqrt{a}$. So we obtain $0<y_1\leq\sqrt{a}$. And since we defined $x_1=\sqrt{a+b\sqrt{a}}, x_1\leq\sqrt{a+b\sqrt{a}}$. So then the theorem is true for $n=1$.

Now we turn to the general case. Suppose Theorem 3 holds for some n. We will show that it holds for $n+1$. Since we have $x_n\leq\sqrt{a+b\sqrt{a}}$, we can use this to get the desired result:

$a-x_n\geq a-\sqrt{a+b\sqrt{a}}\geq 0$.

Since $y_{n+1}=\sqrt{a-b\cdot x_n}$, $y_{n+1}\geq 0$.

Since $b>0, x_n\geq 0$, $y_{n+1}=\sqrt{a-b\cdot x_n}<\sqrt{a-0}=\sqrt{a}$.

We can now use this to show:

$\sqrt{a}=\sqrt{a-0}<\sqrt{a-b\cdot y_n}=x_{n+1}\leq\sqrt{a+b\sqrt{a}}$. This proves Theorem 3 for the general case.

So now we ask the question of what numbers a and b satisfy the conditions of Theorem 3?

(6.1) $a>b\sqrt{a+b\sqrt{a}}$

(6.2) $\frac{a}{b^2}>\frac{\sqrt{a+b\sqrt{a}}}{b}$

(6.3) $(\frac{a}{b^2})^2 - \frac{a}{b^2} > \sqrt{\frac{a}{b^2}}$. Now we set $t = \frac{a}{b^2}$, where $t>0$.

(6.4) $t^2 - t > \sqrt{t}$, $t^2 - t - \sqrt{t} > 0$

We can also see that some c exists such that Equation 6.4 is satisfied if and only if $t>c$. Making the equation an equality rather than an inequality and solving the equation results in $c \approx 1.75488$(Note: We can find an explicit formula for c but for our purposes the approximation will suffice.). We now are prepared to investigate the convergence of the sequences x_n and y_n and what the sequences converge to.

7. Theorem 4 and Its Implications

We will begin by defining $x = \lim_{n \to \infty} x_n$ and $y = \lim_{n \to \infty} y_n$.

THEOREM 4: *Let* c, x, *and* y *be defined as above. For any* $k \geq 0$, *let* a *and* b *be real numbers and satisfy the following:*

1. $0 < b < \frac{2k}{(\sqrt{4c-3}) - 1}$, *and*

2. $a = k^2 + bk + b^2$.

Then, $y = k$ *and* $x = k + b$.

Proof: First a general outline of the proof: In Part I, we will use the conditions to show that the initial conditions of Theorem 3 are satisfied, which will lead us to the conclusion that our functions are well defined and furthermore we will be able to use the results of Theorem 3 in Part II. Part II will show that the sequences converge and what they converge to.

Part I. Let k be given. Now from our above analysis we know that $\frac{a}{b^2} > c \Leftrightarrow a > b\sqrt{a + b\sqrt{a}}$. By Condition 2 $a = k^2 + bk + b^2$. $\frac{a}{b^2} > c \Leftrightarrow (\frac{k}{b})^2 + (\frac{k}{b}) + 1 > c \Leftrightarrow (\frac{k}{b})^2 + (\frac{k}{b}) + 1 - c > 0$. This is true if and only if $(\frac{k}{b}) > x$, where x is the positive root of $x^2 + x + 1 - c = 0$. Using the familiar quadratic equation and taking the positive root we obtain $x = \frac{-1 + \sqrt{4c-3}}{2}$ or $\frac{k}{b} > \frac{-1 + \sqrt{4c-3}}{2}$. We used solely if and only if statements so therefore Conditions 1 and 2 being satisfied is equivalent to the initial condition of Theorem 3, $a > b\sqrt{a + b\sqrt{a}}$, and thus the results of Theorem 3 are applicable.

Part II. Now we will combine the results of Part I and Theorem 3. Since $a > b\sqrt{a + b\sqrt{a}}$, $\sqrt{a} \leq x_n$.

(7.1) $$|y_{n+1} - k| = \frac{|y_{n+1}^2 - k^2|}{|y_{n+1} + k|}$$

(7.2) $\leq \frac{|(a-bx_n)-k^2|}{k}$ (because $y_{n+1}=\sqrt{a-bx_n}>0$)

(7.3) $= \frac{|bk+b^2-bx_n|}{k}$ (Condition 2)

(7.4) $= \frac{b|(k+b)^2-x_n^2|}{k((k+b)+x_n)}$

(7.5) $= \frac{b|(k+b)^2-(a+by_{n-1})|}{k|(k+b)+x_n|}$ (uses Condition 2 again)

(7.6) $= \frac{b|kb-by_{n-1}|}{k|(k+b)+x_n|}$

(7.7) $\leq \frac{b^2|y_{n-1}-k|}{k|(k+b)+\sqrt{a}}$ $(\sqrt{a}\leq x_n)$

(7.8) $= (\frac{k}{b}(\frac{k}{b}+1+\sqrt{\frac{a}{b^2}}))^{-1} \cdot |y_{n-1}-k|$.

Since $\frac{k}{b}>\frac{-1+\sqrt{4c-3}}{2}$, and $c>1.75$, $\frac{k}{b}>\frac{1}{2}$. Furthermore since $\sqrt{\frac{a}{b^2}}=\sqrt{\frac{k^2+bk+b^2}{b^2}}>1$, we can look at the first part of Equation 7.8 as follows: $(\frac{k}{b}(\frac{k}{b}+1+\sqrt{\frac{a}{b^2}}))^{-1}<(0.5(0.5+1+1))^{-1}<1$. Thus, looking back at 7.8 we can conclude that $y=k$. By definition $x_n=\sqrt{a+by_{n-1}}$, so $x=\sqrt{a+kb}=\sqrt{k^2+2bk+b^2}=b+k$. Thus we have proven Theorem 4.

It is clearly apparent that $x\neq y$. As such the limit of our alternating series depends on the initial sign; if we switch all signs around, somewhat unsurprisingly, we obtain a different result. Furthermore if we have a sequence where the first sign alternates, this sequence will never converge to a limit (assuming we choose identical constant values). This is easy to see because we can view the sequence as the union of two disjoint subsequences, x and y, which do not converge to the same limit so clearly the entire sequence can not converge to a limit. Furthermore we can analyze what numbers are possible for our limits of sequences x and y to be:

First off we note Condition 1, $b<\frac{2k}{(\sqrt{4c-3})-1}\approx 1.98k$ (since we know c).

For the sequence y_n, we can choose any positive number k and construct an alternating series that approaches k. The numbers just must satisfy our initial parameters, $b \leq 1.9k$ and $a = k^2 + bk + b^2$. We can use a very similar argument for the x_n's.

8. Conclusion

There are many different routes of exploration available in this field. Some examples the authors give include questioning into the infinitely many possible nested radicals, where each individual sign can be either positive or negative. Clearly any such construction would be larger than that of all negatives and smaller than that of all positive signs. We could also explore into what would happen if we allowed the a's and b's to be sequences rather than constants. Right off the bat we expect that these sequences will be convergent (or at least not go too either infinity) although it is definitely possible to create a sequence that the infinitely nested radical converges yet the sequence does not (for example, let $b_n = (-1)^n$ and this turns into our alternating infinite radicals. Another interesting area would be that of complex numbers and what happens if you allow the constants to be complex (or even negative, which could lead to complex numbers as our limits). There are lots of interesting avenues into which research in this area could head and it is an extremely fascinating area of research.

9. References

[1] C. Ho, S. Zimmerman, *On Infinitely Nested Radicals*, Mathematics Magazine, Feb. 2008.