Self-similarity, or How Even Things Like Cantor Sets and the Golden Ratio are Related

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1. Introduction

At first glance, it might appear that the concept of a Cantor set and the golden ratio are a strange pair to marry; the former is typically presented to wide-eyed undergraduates as a trippy example of an uncountable set with Lebsegue measure zero, while the latter is claimed by people of the new-age variety to be the key to making aesthetically pleasing credit cards and book covers. Nonetheless, the Cantor set and the golden ratio share one important similarity in that both are self-similar, and it is this property that ties the two together in the solution to a clever little geometric problem [1].

2. On Cantor Sets and the Golden Ratio

2.1 The Golden Ratio

The *golden ratio*, often denoted by the letter φ , is equal to $(1 + \sqrt{5})/2$. Of course, this isn't just some random number that some mathematicians decided to give significance to for no reason.

Suppose you have some point a on the interval (0,1). If a is equal to φ, then a divides the interval [0,1] into two intervals such that the ratio of the length of the longer interval and the interval [0,1] is equal to the ratio of the length of the shorter interval to the length of the longer. In other words,

$$\frac{1}{a} = \frac{a}{1-a}$$

Solving this equation for a and taking the positive root in the resulting quadratic equation gives us the value for φ, the golden ratio. The negative root, φ', has the property that $\varphi' = -\varphi^{-1}$. This means that while φ divides the interval [0,1] according to the golden ratio, $1 - \varphi$ will do the same for the interval $[0, \varphi]$. This is a key example of the self-replicating property of φ.

2.2 The Cantor Set

Most are familiar with the Cantor ternary set, which is created by deleting the open middle third of each segment at each step. To construct this set, start with the interval [0,1], then delete the segment $({}^{1}/_{3}, {}^{2}/_{3})$, leaving you with the two segments $[0, {}^{1}/_{3}]$ and $[{}^{2}/_{3}, 1]$. Repeat this process, removing the open third of each remaining segment.

Here, we will be more concerned with the *middle-α Cantor sets*. These sets are constructed in much the same way that the Cantor ternary set is. First, choose some number $\alpha \in (0,1)$, then remove the open interval of length α from the middle [0,1]. You will be left with two intervals of length $(1 - \alpha)/2$. Call this quantity β. Notice that β can not be longer than half the length of the original interval, thus $\beta \in (0, {}^{1}/_{2})$. After this, remove open intervals of length $\alpha\beta$ from each interval, leaving you with four intervals of length β^2. Continue this process.

After taking n steps, your set will consist of a union of 2^n intervals, each of length β^n. Call this union I_n. Thus, the middle-α Cantor set on the interval [0,1] is:

$$C_\alpha \equiv \bigcap_{n=0}^{\infty} I_n$$

The middle-α Cantor set exhibits self-similarity like the golden ratio, in that β is a scaling factor for this set. This means that if you take the set on the interval $[0,\beta]$ and scale it up to the unit interval, you will have an exact replica of the original set. This means that all integer powers of β are likewise scaling factors.

3. Self-similarity and Cleverness

Cantor sets also have the interesting property that they are completely discontinuous, meaning that between any two points in the set there are points not in the set. This leads to an interesting geometric problem:

Given β ∈ (0, $^1/_2$), is it possible to find a λ ∈ (0,1) such that $C_\alpha + \lambda C_\alpha = \{0\}$?

In other words, is it possible to have two Cantor sets that only share a point at the origin, and all of whose other points fall into the holes of the other set? As it turns out, you can, for small enough β, that is, β < $(3 - \sqrt{5})/2$, a number that looks conspicuously similar to the golden ratio.

In order to find a β and λ that satisfy the above condition, we only need to find a β and λ that satisfy this condition for I_1 (that is, so $I_1 + \lambda I_1 = \{0\}$). This is because, as previously discussed, the Cantor set is self-similar, and thus all the I_ns for $n > 1$ are simply I_1, scaled down and repeated. This leads us to the solution:

$$\frac{\beta}{1-\beta} < \lambda < 1 - \beta$$

Solving this set of inequalities leads us to our solution, β < $(3 - \sqrt{5})/2$, while no solution can exist when $\beta \geq (3 - \sqrt{5})/2$. To see this fact, we will investigate the intersection of C_α with a shrunken and translated copy of itself.

Suppose $\beta \in [1/3, 1/2)$, and let $\alpha = 1 - 2\beta$. Then, if $\lambda \in [\alpha, 1]$, then for any $t \in [-l, 1]$, $C_\alpha \cap (\lambda C_\alpha + t) \neq \{0\}$. To prove this, first we show that $I_1 \cap (\lambda I_1 + t) \neq \{0\}$. We know this to be true because, since $\lambda \geq \alpha$, $\lambda I_1 + t$ can not be contained in the open gap in I_1. Further, since $\beta \geq 1/3$ and $\lambda \leq 1$, we know that $\beta \geq \lambda\alpha$. Thus, I_1 can not be contained in the open gap in $\lambda I_1 + t$. Thus, $I_1 \cap (\lambda I_1 + t) \neq \{0\}$. Now, suppose $I_n \cap (\lambda I_n + t) \neq \{0\}$. Let J_n and J_n' denote any components of I_1 and $\lambda I_1 + t$ that intersect. Then $(J_n \cap I_{n+1}) \cap (J_n' \cap (\lambda I_{n+1} + t)) \neq \{0\}$. Thus, through induction, $C_\alpha \cap (\lambda C_\alpha + t) \neq \{0\}$.

Next, we want to show that if $\beta \in [(3-\sqrt{5})/2$, 1/2) and λ ∈ $(0,1)$, then $C_\alpha \cap \lambda C_\alpha \neq \{0\}$. First we show that this is true if λ ∈ $[\beta, 1]$. Let J and J' be intersecting components of I_1 and λI_1, such that $J \cap J'$ does not contain zero. Then there is an affine map that maps the segment of C_α contained in J onto C_α and the segment of λC_α contained in J' onto $\lambda C_\alpha + t$. However, we've just proven that $C_\alpha \cap (\lambda C_\alpha + t) \neq \{0\}$ when $\beta \in$ [1/3, 1/2), so, since $(3-\sqrt{5})/2 >$ 1/3, $C_\alpha \cap \lambda C_\alpha \neq \{0\}$. Now that we know this to be true for λ ∈ $[\beta, 1]$, suppose that λ ∈ $[\beta^{n+1}, \beta^n)$. We can use a similar mapping as before so that $\lambda/\beta^n \in [\beta, 1)$. Thus, we conclude that, when $\beta \geq (3-\sqrt{5})/2$ and λ ∈ $(0,1)$, $C_\alpha \cap \lambda C_\alpha$ contains a non-zero point, making a solution to our problem impossible.

Now, while this solution does look strikingly similar to the golden ratio, it's not quite it. So what does the golden ratio have to do with this? We see that when we take the ratio of β to α, when β = $(3-\sqrt{5})/2$:

$$\frac{\beta}{\alpha} = \frac{\beta}{1-2\beta} = \frac{(3-\sqrt{5})/2}{1-3-\sqrt{5}} = \frac{1+\sqrt{5}}{2}$$

There it is! This means that $1-\beta$ divides the interval [0,1] according to the golden ratio, and it is below this value of β that we have a solution to our geometric problem.

The ratio β/α is referred to as the *thickness* of a middle-a Cantor set, and we find that φ is the critical point above which the set is too thick for a solution to be possible.

4. Conclusion

It might be a little funny how often φ makes an appearance in our daily lives, but we find that one of its many properties it its self-similarity, and it is this self-similarity that ties φ to the thickness of Cantor sets. In addition to making this clever observation, the paper shows how self-similarity can be used to simplify an existing problem, a method that can no doubt be applied to many other things in math.

5. References

[1]: Roger L. Kraft, *A Golden Cantor Set*, The American Mathematical Monthly, Vol. 105, No. 8 (Oct., 1998), pp. 718-725.