Differentiating Under the Integral Sign

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1 Introduction

The goal of this paper is to discuss the process of differentiating under the integral sign. Specifically, we review a short journal article **[3]**, which discusses when this interchange of limiting operations is valid by providing a set of necessary and sufficient conditions for its validity.

The most commonly discussed (and most commonly used) sufficient condition for differentiation under the integral sign is that the integral of the derivative of a function converges uniformly. However, the conditions proven here are dependent on the integrability of all derivatives of the function in question; as a result of this, Riemann or Lebesgue integrals (for which no guarantee is made regarding the integrability of derivatives) will not suffice for this set of conditions. As a result, the Henstock integral, which provides an extremely strong definition of the Fundamental Theorem of Calculus is used; what follows in this paper is an introduction to the Henstock integral and other terminology used in the paper, a summary of the theorems exhibited in the paper, and an in-depth analysis of the main theorem (which ultimately proves the aforementioned conditions). Furthermore, we will also provide further discussion of the material presented in the paper, with particular emphasis on the Henstock integral and its properties.

2 Terminology

2.1 Henstock Integration

We consider a function $f : [-\infty, \infty] \to (-\infty, \infty)$.

- We consider an *open interval* in $[-\infty, \infty]$ to be of the form (a, b), $[-\infty, b)$, $(a, \infty]$, or $[-\infty, \infty]$, where $-\infty \le a < b \le \infty$.
- A *gauge* is a mapping γ such that, for all $x \in [-\infty, \infty]$, $\gamma(x)$ is an open interval containing x.
- A *tagged partition* is a finite set of pairs $(z_i, I_i)_{i=1}^N$ such that each I_i is closed, $z_i \in I_i$, $I_i \cup I_j$ consists of at most a single point if $i \ne j$, and $\cup_{i=1}^N I_i = [-\infty, \infty]$.
- A tagged partition is γ-*fine* if, given a gauge γ, $I_i \subset \gamma(z_i)$ for all i.
- We then consider f to be **Henstock integrable**, and we write that $\int_{-\infty}^{\infty} f = A$, if there is real A such that, for all $\epsilon > 0$, there is a gauge γ such that, for any γ-fine tagged partition of $[-\infty, \infty]$, then:

$$\left| \sum_{i=1}^{N} f(z_i) \, |I_i| - A \right| < \epsilon$$

(Note that, if I is unbounded, we take $|I| = 0$.)

- Furthermore, we take the integral of f over an interval $[a, b]$ to be:

$$\int_{-\infty}^{\infty} f \cdot \chi_{[a,b]}$$

where $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$.

2.2 Additional Terms

Let f be a real-valued function on $[a, b]$.

- We say that f is *absolutely continuous* (AC) on a set $E \subset [a, b]$ if, for any $\epsilon > 0$, there is a $\delta > 0$ such that, given any finite set of disjoint open intervals $(x_i, y_i)_{i=1}^N$ such that $x_i, y_i \in E$ and $\sum_{i=1}^N (y_i - x_i) < \delta$, we have $\sum_{i=1}^N (f(y_i) - f(x_i)) < \epsilon$.
- Furthermore, we say that f is *absolutely continuous in the restricted sense* (AC_*) if, given the same conditions as above, we have:

$$\sum_{i=1}^N \sup_{x,y \in [x_i, y_i]} |f(y) - f(x)| < \epsilon.$$

Note that any AC_* function is AC.

- Finally, we say that f is *generalized absolutely continuous in the restricted sense* (ACG_*) if f is continuous and E is the union of all sets on which f is AC_*. Note that the ACG_* functions are properly contained in the set of functions differentiable almost everywhere and that they contain the set of functions differentiable nearly everywhere (namely, differentiable except perhaps on a countable set).

3 Intermediate Results

We now present the intermediate results discussed in the paper. The final result is discussed in much greater detail in the next section.

Theorem 1. Let f be a real-valued function on $[a, b] \subset [-\infty, \infty]$. Then $\int_a^b f$ exists and equals (real-valued) A if and only if f is integrable on each subinterval $[a, x] \subset [a, b]$ and $\lim_{x \to b-} \int_a^x f$ exists and equals A.

This theorem basically represents the fact that Henstock integrals have no concept of an "improper integral" (as is found in, say, Riemann integration). This is briefly explained by the fact that, under Henstock integration, the length of an unbounded interval is taken as zero, an assumption not dissimilar to those made for improper Riemann integrals. Essentially, the theorem provides that any Henstock-integrable function may be integrated with a proper Henstock integral (since it would be integral on each subinterval, as described above).

The next theorem is a restatement of the Fundamental Theorem of Calculus that uses the Henstock integral. Note that it is a significantly stronger result than the Fundamental Theorem of Calculus for Riemann integrals.

Theorem 2.

1. Let f be a real-valued function on $[a,b]$. Then $\int_a^b f$ exists and $F(x) = \int_a^x f$ for all $x \in [a,b]$ if and only if F is ACG_* on $[a,b]$, $F(a) = 0$, and $F' = f$ almost everywhere on (a,b).
2. Let F be a real-valued function on $[a,b]$. Then F is ACG_* if and only if F' exists almost everywhere on (a,b), F' is Henstock integrable on $[a,b]$, and $\int_a^x F' = F(x) - F(a)$ for all $x \in [a,b]$.

Finally, we have a corollary of the Fundamental Theorem that provides a sufficient condition for asserting the derivative of a function to be integrable.

Corollary 3. Let F, a real-valued and continuous function on $[a,b]$, be differentiable nearly everywhere on (a,b). Then F' is Henstock integrable on $[a,b]$ and $\int_a^x F' = F(x) - F(a)$ for all $x \in [a,b]$.

Of course, this follows immediately from Theorem 2 and the facts about ACG_* functions stated above (namely, that any function differentiable nearly everywhere is ACG_*).

These results (mainly the Fundamental Theorem of Calculus) are used to prove the theorem (and corollaries) that provide the conditions for differentiation under the integral sign. The statement and proof of this theorem follow.

4 Primary Result

Theorem 4. Let f be a real-valued function on the region $[\alpha,\beta]\times[a,b]$. Suppose that $f(x,y)$ is ACG_* as a function of x on $[\alpha,\beta]$ for almost all $y \in (a,b)$. Then, letting $F(x) = \int_a^b f(x,y)dy$, we have that F is ACG_* on $[\alpha,\beta]$ and $F'(x) = \int_a^b f_x(x,y)dy$ for almost all $x \in (\alpha,\beta)$ if and only if, for all $[s,t] \subset [\alpha,\beta]$:

$$\int_{x=s}^{t} \int_{y=a}^{b} f_x(x,y)dydx = \int_{y=a}^{b} \int_{x=s}^{t} f_x(x,y)dxdy.$$

Proof. First, suppose F is ACG_* and that $F'(x) = \int_a^b f_x(x,y)dy$. We apply the second part of the Fundamental Theorem of Calculus twice, as follows:

$$\int_{x=s}^{t}\int_{y=a}^{b} f_x(x,y)dydx = \int_{x=s}^{t} F'(x)dx = F(t) - F(s)$$

$$= \int_{y=a}^{b} (f(t,y) - f(s,y))dy = \int_{y=a}^{b}\int_{x=s}^{t} f_x(x,y)dxdy.$$

To prove the converse, first let $x \in (\alpha, \beta)$ and let h be a real number such that $x + h \in (\alpha, \beta)$. We again apply the second part of the Fundamental Theorem as follows:

$$\int_{\widetilde{x}=x}^{x+h}\int_{y=a}^{b} f_x(\widetilde{x},y)dyd\widetilde{x} = \int_{y=a}^{b}\int_{\widetilde{x}=x}^{x+h} f_x(\widetilde{x},y)d\widetilde{x}dy = \int_{y=a}^{b} (f(x+h,y) - f(x,y))dy$$

$$= \int_{y=a}^{b} f(x+h,y)dy - \int_{y=a}^{b} f(x,y)dy = F(x+h) - F(x).$$

Also, by the definition of limits:

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_{\widetilde{x}=x}^{x+h}\int_{y=a}^{b} f_x(\widetilde{x},y)dyd\widetilde{x}.$$

But, by the first part of the Fundamental Theorem, this is equal to $\int_a^b f_x(x,y)dy$ for almost all $x \in (\alpha, \beta)$. Also, we have:

$$\int_{\alpha}^{x} F' = \int_{\widetilde{x}=\alpha}^{x}\int_{y=a}^{b} f_x(\widetilde{x},y)dyd\widetilde{x} = \int_{\widetilde{x}=\alpha}^{x} F'(\widetilde{x})dx = F(x) - F(\alpha)$$

This holds for all $x \in [\alpha, \beta]$; thus, F is ACG_* on $[\alpha, \beta]$, completing the argument.

5 Corollaries

This primary theorem has four corollaries, each of which provide additional insight into interchanges of operations. The first is a generalization of the above theorem, relying on the fact that only the linearity of the integral over $y \in [a, b]$ was employed. The next three, however, provide insight into various interchanges of operations, and will thus be briefly discussed below.

Corollary 6. Let g be a real-valued function on the region $[\alpha, \beta] \times [a, b]$. Suppose that $g(x, y)$ is integrable over $[\alpha, \beta]$ for almost all $y \in (a, b)$. Letting $G(x) = \int_a^b \int_\alpha^x g(\widetilde{x}, y)d\widetilde{x}dy$, we have that G is ACG_* on $[\alpha, \beta]$ and $G'(x) = \int_a^b g(x, y)dy$ for almost all $x \in (\alpha, \beta)$ if and only if, for all $[s, t] \subset [\alpha, \beta]$:

$$\int_{x=s}^{t}\int_{y=a}^{b} g(x,y)dydx = \int_{y=a}^{b}\int_{x=s}^{t} g(x,y)dxdy.$$

This corollary provides necessary and sufficient conditions for interchanging iterated integrals. The next, by combining the previous two, will provide necessary and sufficient conditions for interchanging summation and integration.

Corollary 7. Let $g_n(x)$ be a sequence of functions defined for $x \in [\alpha, \beta]$ and natural n. Suppose that g_n is integrable over $[\alpha, \beta]$ for each natural n/ Letting $G(x) = \sum_{n=1}^{\infty} \int_{\alpha}^{x} g_n(\widetilde{x})d\widetilde{x}$, we have that G is ACG_* on $[\alpha, \beta]$ and $G'(x) = \sum_{n=1}^{\infty} g_n(x)$ for almost all $x \in (\alpha, \beta)$ if and only if, for all $[s,t] \subset [\alpha, \beta]$:

$$\int_{x=s}^{t}\sum_{n=1}^{\infty} g_n(x)dx = \sum_{n=1}^{\infty}\int_{x=s}^{t} g_n(x)dx.$$

Finally, we have the conditions for differentiating under the integral sign, which are stated more precisely here.

Corollary 8. Let f be a real-valued function on the region $[\alpha, \beta] \times [a, b]$.

1. Suppose that $f(x,y)$ is continuous on $[\alpha, \beta]$ for almost all $y \in (a,b)$ and differentiable nearly everywhere in (α, β) for almost all $y \in (a,b)$. If $\int_{x=s}^{t}\int_{y=a}^{b} f_x(x,y)dydx = \int_{y=a}^{b}\int_{x=s}^{t} f_x(x,y)dxdy$, then $F'(x) = \int_a^b f_x(x,y)dy$ for almost all $x \in (\alpha, \beta)$.

2. Suppose that $f(x,y)$ is ACG_* on $[\alpha, \beta]$ for almost all $y \in (a,b)$ and that $\int_a^b f_x(x,y)dy$ is continuous on $[\alpha, \beta]$. If $\int_{x=s}^{t}\int_{y=a}^{b} f_x(x,y)dydx = \int_{y=a}^{b}\int_{x=s}^{t} f_x(x,y)dxdy$, then $F'(x) = \int_a^b f_x(x,y)dy$ for all $x \in (\alpha, \beta)$.

6 Additional Discussion

We now provide additional insight into the following topics discussed in the paper.

6.1 The Henstock Integral and Integration Theory

Although the concept of integration was not founded until the invention of calculus (17th c.), the motivation has been present since the ancient Greek problem of quadrature (finding a square with the same area as a given plane figure) **[2]**. The first rigorous definition of an integral, however, was given by Cauchy in the 1820s and revised by Riemann into what we now call the Riemann integration theory. Further integration theories served to build on the Riemann integral and strive for a more "complete" integral (one which would allow the integration of a greater set of functions).

The first major revision to integration theory was given by Lebesgue in the early 20th century. Lebesgue integration replaced the Jordan measure employed by the Riemann integral with a different measure theory (the Lebesgue measure). The Lebesgue measure relaxes the restrictions on measurability (for instance, requiring only countable partitions rather than finite partitions when computing the measure of a set), which vastly increases the number of measurable sets and thus the number of integrable functions in comparison to the Jordan measure.

The Henstock integral (also called the HK-integral) was devised in 1955 (and also thought of by Kurzweil two years later, hence the alternate name). It provides a definition of integration (repeated above) that requires no measure theory to formulate. Furthermore, the Henstock integral has three interesting properties, two of which relate it to Riemann and Lebesgue integrals:

- The Henstock integral of a function is uniquely defined when it exists.
- A function is Riemann integrable if and only if it is Henstock integrable and the gauge γ in the definition of Henstock integrability can be chosen to be constant. (Notably, every Riemann integrable function is Henstock integrable.)
- Every Lebesgue integrable function is Henstock integrable. In addition, the two integrals will have the same values.

The Henstock integral is thus at least as "complete" as its predecessors; however, it is far from being a a "perfect integral" (i.e. one that would make every function integrable). In order for such a theory to exist, there would need to be such a thing as a *total measure* of the reals - that is, there would need to be a measure theory which assigns a measure to every subset of the reals. However, the prospects of finding such a measure theory are poor, because:

- It has been shown (by Vitali) that such a measure cannot be both countably additive and translation invariant. (Generally, in searching for such a measure, countable additivity is retained.)
- It has also been shown that, if there is a countably additive total measure of the reals, then set theory is consistent.
- But Gödel's second incompleteness theorem states that set theory cannot prove its own consistency. And so the existence of such a total measure, and thus that of a "perfect" integral, is not provable. (However, it has not been proven that a perfect integral *cannot* exist.)

6.2 Interchanging Limiting Operations

The paper reviewed provided theorems which can be used to justify interchange of integration and several different limiting operations; specific cases were given for integration, differentiation, and summation. Such criteria are of great importance, since attempting to interchange such operations when the criteria are not met can result in completely incorrect results (as observed by the paper, Cauchy's violation of these conditions resulted in his devising a convergent value for a divergent integral, equating $\int_0^\infty x \sin x^2 \sin sx dx$ to $\frac{s}{4}\sqrt{\frac{\pi}{2}}(\sin \frac{s^2}{4} + \cos \frac{s^2}{4})$ through incorrect differentiation of a valid integral).

The theorems given apply Henstock's theory of integration, but it should be noted that similar results can apply to the well-known uniform convergence condition for Riemann integrals; as noted in the introduction, this is a sufficient condition for differentiation under the integral sign. However, as Riemann-integrable functions are necessarily Henstock integrable (see above), the only advantage provided by this condition is simplicity, while this comes at the cost of precision (in that not all Riemann-integrable functions are Henstock-integrable, and in that no necessary condition has been precisely stated for Riemann integrals). The Henstock-integral conditions are best for these purposes, hence their usage in the reviewed paper.

It should also be briefly noted that other interchanges of operations (such as differentiating or summing under a summation sign) can be justified with such criteria. However, being outside the scope of the reviewed paper, they will not be discussed here in any detail.

7 References

[1] G. Folland, Real analysis: modern techniques and their applications, 2nd ed.(1999) pp. 43-84.

[2] A. Shenitzer and J. Steprans, The Evolution of Integration, *Amer. Math. Monthly*, **101**(1994) pp. 66-72.

[3] E. Talvila, Necessary and Sufficient Conditions for Differentiating under the Integral Sign, *Amer. Math. Monthly*, **108**(2001) pp. 544-548.