Option Pricing

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1 Introduction

In finance, there are many types of transactions. Your everday cash transactions, credit card purchases, even bartering with tangibles is considered a financial transacion. There are other more specialized financial activities such as buying and selling of stock and bonds. Particularly in the corporate world, more financial transactions have been developed that allow for various gains (and losses) on these standard forms of trading. Some of these unusual trading techniques are referred to as financial derivatives. Options, a type of financial derivative, provides an alternative approach to buying assets. Essentially, the buyer of the option has the right to purchase assets, usually stock, at a predetermined price, usually lower than the market price, while the seller has the obligation to sell the asset. This transaction, if executed, is to be completed on an expiration date. Adding in all of these variables makes it difficult to determine the value of an option at a specific point in time. As the theoretical value of options change over time, the risks associated with buying and selling options changes too. In an effort to approximate risk levels and values of portfolios, it is important in the financial realm to approximate the value of options. Option valuation is also a useful quantifier of corporate liabilities, since the majority of corporate liabilities can be described by options. Mathematical models using risk neutral pricing and stochastic calculus are used in this valuation process.

There are many models that have been developed to asses the value of options. They all take various approaches and hold some assumptions constant, but they do manage to somewhat accurately predict a very random process. Black and Scholes in 1973 produced the most prominent method for valuing options. By assuming that stock prices follow a lognormal distribution, they develop a differential equation that governs the value of the option [2]. Others have followed by using a series of difference equations in place of Black and Scholes differential equation. A second approach that has been explored by Boyle is a Monte Carlo simulation. This approach is more flexible in that it can adapt to different processes that determine the underlying stock returns [2]. Merton, another financial researcher, has recently proposed a model that considers continuous pieces and jump pieces to account for the arrival of pertinent information which affects the underlying stock price.

Options are a complex method of buying and selling assets. If executed improperly, the risks are great and the losses can be significant. If understood, options can have lower risk than stock, yet they potentially can have higher return rates. Options can also be used as hedge positions, where they are used to protect against future losses. As investors have begun to understand options' complexity and importance of options they have become a more common financial transaction. Thus, there has been a lot of recent research on the simulation and modeling of option pricing. These papers are a result of researchers analyzing various approaches to valueing options. Through the literature, it is evident that the models that are developed build upon the existing models in efforts to perfect and generalize the results.

2 Terminology

2.1 Financial

- **Asset:** Anything tangible or intangible that will produce future cash flows for a company.
- **Option:** A type of financial derivative where an asset is bought or sold between two parties at a predetermined price within a specified time period. The the buyer has the right to the transaction, while the seller has the obligation to fulfill the exchange.
- **European Option:** The option transaction must be completed on the expiration date (maturity date).
- **American Option:** The option transaction may take place any day, as long as it is by the expiration date (maturity date).
- **Call Option:** The right to buy a single share of common stock.
- **Exercise Price or Striking Price:** The predetermined price of the option, and it is the price that is paid when the option is exercised.
- **Value of an Option:** The value of an option is determined by the difference between the strike price (exercise price) and the market price of the underlying asset.
- **Hedge Position:** Financial transactions that protect against possible losses.
- **Short Selling:** Short selling is taking the risk that an asset market price will drop, so the short seller borrows an asset, such as a number of shares of stock, from Company A, and sells it at market price. When the market price drops, the short seller repurchases the stock at the lower amount. Then, they deliver the stock, keeping a profit.
- **Going Long:** Going long is the opposite of short selling. It takes the assumption that an asset market price will rise, so the holder of the long position borrows an asset, such as a number of stock, from Company A, and sells it after the market price increases. Once returning the borrowed stock at the original price, the holder of the long has gained a profit.
- **Derivative:** A derivative is a financial contract based on underlying variables, such as stock shares. The underlying variables can range drastically from tangible assets to intangible financial indexes.

3 Valuation Models

3.1 Black-Scholes Model

In previous valuation models, the approach was to find the equilibrium between the expected return on a hedge position and the return on a riskless asset. The point at which these are equal is taken to be the initial condition when deriving the differential equation of the Black-Scholes Model.

Assumptions:

- Short term interest rate is constant and known.
- Stock prices follow a random walk, and the distribution over a finite time interval is lognormal. Thus, the variance rate of return on the stock is held constant.
- The buyer or seller is allowed to borrow any fraction of the option to buy or hold it at the short-term interest rate.
- European options; exercised at maturity.
- No penalties for short selling.

Thus, the value of the option given these assumptions strictly depends on the price of the underlying stock and the duration until maturity.

Proof. Taking $w_x(x,t)$ to be the value of the option, where x is the stock price and t is the time. Since the Black-Scholes Model is derived based on the fact that the hedge position must equal the return on a riskless asset, so we can define the number of options that need to be short versus one share of long is:

$$1/w_x(x,t) \tag{1}$$

Since the hedge position is being equated to the combination of long and short positions, the value of the equity in position is:

$$\Delta x - \Delta w/w_x \tag{2}$$

Then, by using stochastic calculus, including Brownian Motion and Ito's Lemma, we expand Δw, and rearrange to get:

$$\Delta w = w_x \Delta x + \frac{1}{2} w_{xx} v^2 x^2 \Delta t + w_2 \Delta t \tag{3}$$

In equation (3) above, v^2 is the variance rate of return on the stock. This variance value is held constant as indicated in the assumptions. Also, v is the volatility of the underlying stock, which is the indicator of how the stock price varies. Now, by substituting equation (3) into expression (2), we get:

$$-(\frac{1}{2} w_{xx} v^2 x^2 + w_t)\Delta t/w_x \tag{4}$$

This is the change in the value of the equity in the hedge position. This change in the value must equal the value of the equity times $r\Delta t$, because we know the return on the equity in the hedge position is certain, and the return must equal $r\Delta t$. So, equating these, we get:

$$-(\frac{1}{2}w_{xx}v^2x^2 + w_y)\Delta t/w_x = (x - w/w_x)r\Delta t \tag{5}$$

Now, rearranging, we get the differential equation for the value of the option:

$$w_y = rw - rxw_x - \frac{1}{2}v^2x^2w_{xx} \tag{6}$$

Using $t*$ as the maturity date for the option, and c as the exercise price, we know that the value of the option at maturity is the stock price less the exercise price. If the stock price is less than the exercise price, then the option has value zero.

$$w(x, t_m) = \begin{cases} x - c & \text{if } x \geq c, \\ 0 & \text{if } x < c. \end{cases} \tag{7}$$

Taking this to be our boundary condition, the following substition leads us to solve the differential equation in (6).

$$\begin{aligned} w(x,t) = e^{r(t-t_m)}y\Bigg[&(2/v^2)\Big(r - \frac{1}{2}v^2\Big) \\ &\Big[\ lnx/c - (r - \frac{1}{2}v^2)(t - t_m)\Big], \\ &- (2/v^2)\Big(r - \frac{1}{2}v^2\Big)^2(t - t_m)\Bigg]. \end{aligned} \tag{8}$$

We find that the differential equation has solution:

$$w(x,t) = xN(d_1) - ce^{r(t-t_m)}N(d_2) \tag{9}$$

$$d_1 = \frac{lnx/c + (r + \frac{1}{2}v^2)(t_m - t)}{v\sqrt{t_m - t}} \tag{10}$$

$$d_2 = \frac{lnx/c + (r - \frac{1}{2}v^2)(t_m - t)}{v\sqrt{t_m - t}} \tag{11}$$

Where $N(d)$ is the cumulative distribution function of the standard normal distribution, which we initially determined to be logarithmic. This is the Black-Scholes Formula, and an interesting result that can be seen from it is that $w_x(x,t) = N(d_1)$, where the partial derivative of the option value function $w(x,t)$ determines the ratio of stock shares to options that is found in the hedge position [1].

Although the Black-Scholes formula is widely used and is a good model for option values, it uses many unrealistic assumptions, such as no dividend payout, and a constant interest rate. The Monte Carlo simulation provides an alternative approach to valuation problems that takes into consideration more variables associated with the underlying stock. □

3.2 Monte Carlo Simulation

The Monte Carlo simulation provides many advantages to option valuation, including flexibility, modifiability, and adaptability to special situations. A Monte Carlo simulation is used in many fields beyond finances. It measures the uncertainty of a particular value based on the variables determining it. Then the model finds an average value of what it is measuring based on these uncertainties. A basic description of the Monte Carlo method is below [2]:

Consider the integral:

$$\int_A g(y)f(y)\,dy = \bar{g}, \tag{12}$$

Where $g(y)$ is an arbitrary function, and $f(y)$ is a probability density function with the integral over A of $f(y)$ is one. A is the range of integration. Since $f(y)$ is the probability density function that determines the distribution of the rate of return on the stock, it is easy to adjust this function to reflect different probability densities. This is a key element that makes the Monte Carlo approach more flexible to various real world situations.

We find an estimate for the mean of $g(y)$, $\bar{g}$, by summing a sample of size n of random values from $g(y)$.

$$\bar{g} = \frac{1}{n}\sum_{i=1}^{n} g(y_i) \tag{13}$$

And, the standard deviation is:

$$\hat{s}^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(g(y_i) - \bar{g})^2 \tag{14}$$

This statistical simulation is then applied to values attained from many simulations of the Black-Scholes equation to locate a more accurate value for the option. We define the following variables similarly to the Black-Scholes method.

Variables:

- s_t = the stock price at time t
- r = the risk-free interest rate per quarter, compounded continuously
- v^2 = the variance rate of return of the underlying stock, assumed constant

- D_t = the dividend payable at time t, assumed to be paid quarterly
- c = exercise price of the option
- t_m = expiration date of the option

It is also assumed that the rate of return on stock is equal to risk-free interest rate r. Similar to the Black-Scholes method, it is assumed that stock prices follow a lognormal distribution [2].

As is seen simply by the variables, more adjustments can be made using the Monte Carlo approach as opposed to only using the Black-Scholes model. For example, the stock price at various times is considered, where in Black-Scholes it is taken to be constant. Also, the Black-Scholes model doesn't allow for dividend payments, where as the Monte Carlo approach does permit such variabilities.

By combining these approaches, and running many simulations of the Black-Scholes valuation formula under the Monte Carlo method we obtain more accurate results, but there are still more considerations that better model the real world movement of option values. Merton, has ventured into some of these modifications.

3.2.1 Merton's Model

One motivation for Merton to explore modifications to the Monte Carlo method is that the Monte Carlo method's largest downside is that the accuracy of the estimate requires a large number of simulations, since the standard error of the estimates are inversly proportional to the square root of the number of simulations. This is a very inefficient way to arrive at an estimate with a reasonable error range. Efficiency in option valuing is essential as split second decisions need to be made to capitalize on advantageous stock prices. Merton attempts to improve this by considering stronger representation of the underlying stock value. Merton takes stock returns and splits them into a continuous part and a jump component. The idea behind this is that the model can account for events in time that cause significant changes in stock prices. By including the jump component, these jumps in stock returns can be considered in the Black-Scholes equation, and thus improve Monte Carlo simulations. The continuous portion of the stock price uses a Gauss-Weiner process, while the jump component uses an independent Poisson process. The following is the equation for the stock price, considering both continuous and jump components [3]:

Variables:

- s = the stock price
- α = the expected instantaneous return on the stock
- λ = the expected number of arrivals of information that changes stock price per unit time

- t = unit of time
- σ^2 = the instantaneous variance of the return conditional on the Poisson event not occurring
- dz = a Gauss-Weiner process
- $q(t)$ = the Poisson process

The equation is:

$$\frac{ds}{s} = (\alpha - \lambda k)dt + \sigma dz + dq \tag{15}$$

4 Conclusion

As is seen throughout the paper, there are many approaches that can be taken to value options. Option valuation requires difficult calculations due to the large number of unknown variables which are not easily predicted. Through this paper, the classic valuation model produced by Black and Scholes was presented. Some of the assumptions used to derive the Black-Scholes equation simplify the model, but are unrealistic, thus not creating optimum results. This inaccuracy drove the use of a Monte Carlo simulation to help reduce the error when valuing options by running many trials. Many simulations allowed for more variable fluxuation, making the Monte Carlo approach more flexible and adaptable to real world situations. Merton, enhanced this model to achieve an even cleaner approximation. He took the stock price and modeled it as having both a continuous and a jump part. Including a jump component represented a spike or drop in the stock price upon the arrival of new information in time. All of the models discussed above were focusing on European models, where the option is exercised on the maturity date.

American options, on the other hand, don't have such tools to calculate their value. American options, can be exercised any date prior to the exercise date. This adds a level of ambiguity that has yet to be modeled. There are ideas of using Merton's approach of a continous and a jump piece to model American options, because the jump component when a significant event changes the stock price could be used to predict if the option was exercised. Essentially, it is fairly reasonable to approximate if an option transaction will be completed depending on the stock price, although this is still tricky to accurately quantify human motivation behind transaction completion.

Aside from modeling American options, other types of options still need more valuation tools. As mentioned in the beginning, any type of asset can be traded in an option transaction. This paper focused on call options, which are the trading of common stock. The stock return can be explicity determined at a point in time, where as other assets don't have this luxury. This makes modeling of these other forms of options even more difficult. But, with time and exploration into these more focused fields, more direct models will be developed and discovered to value options.

5 References

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