Cauchy-Riemann equations

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Definition 1. Let \( f(z) \) be a complex valued function defined in a neighborhood of a point \( z_1 = x_1 + iy_1 \). Then \( f \) is complex differentiable at \( z_1 \) if

\[
f(z) = f(z_1) + p(z)(z - z_1),
\]

where \( p \) is continuous at \( z_1 \). We define the complex derivative of \( f \) at \( z_1 \) to be the value of \( p \) at \( z_1 \).

\[
f'(z_1) = p(z_1).
\]

Theorem 1. Let \( f(z) = u(x, y) + iv(x, y) \). Then \( f \) is complex differentiable at \( z_1 \) if and only if \( u \) and \( v \) are real differentiable at \((x_1, y_1)\) and

\[
u_x(x_1, y_1) = v_y(x_1, y_1), \quad u_y(x_1, y_1) = -v_x(x_1, y_1).
\]

These are called the Cauchy-Riemann equations.

Proof. Let us write equation (1) in matrix form:

\[
\begin{bmatrix}
u(x, y) \\
v(x, y)
\end{bmatrix} = \begin{bmatrix}
u(x_1, y_1) \\
v(x_1, y_1)
\end{bmatrix} + \begin{bmatrix}a(x, y) & -b(x, y) \end{bmatrix} \begin{bmatrix}x - x_1 \\ y - y_1\end{bmatrix}
\]

We can write this line by line

\[
u(x, y) = u(x_1, y_1) + a(x, y)(x - x_1) - b(x, y)(y - y_1)
\]

\[
v(x, y) = v(x_1, y_1) + b(x, y)(x - x_1) + a(x, y)(y - y_1).
\]

In this form we see that \( u \) and \( v \) are real differentiable at \((x_1, y_1)\) and \( u_x = v_y, \; v_y = -v_x \) at \((x_1, y_1)\).

Next suppose \( u \) and \( v \) are real differentiable at \((x_1, y_1)\) and (3) is true. Let \( a_1 = u_x(x_1, y_1) = v_y(x_1, y_1), \; b_1 = v_x(x_1, y_1) = -u_y(x_1, y_1) \). Let’s write the definition of real differentiability as follows

\[
u(x, y) = u(x_1, y_1) + a_1(x - x_1) - b_1(y - y_1) + \epsilon(z, y)
\]

\[
v(x, y) = v(x_1, y_1) + b_1(x - x_1) + a_1(y - y_1) + d(x, y),
\]

where \( \epsilon(x, y)/|z - z_1| \to 0, \; d(x, y)/|z - z_1| \to 0 \) as \( z \to z_1 \). So

\[
f(z) = f(z_1) + (a_1 + ib_1)(z - z_1) + \epsilon(z)(z - z_1), \quad \text{where} \ \epsilon(z) = \frac{\epsilon(z) + id(z)}{z - z_1}.
\]

and \( \epsilon(z) \to 0 \) as \( z \to z_1 \). This can be rewritten as

\[
f(z) = f(z_1) + p(z)(z - z_1),
\]

where \( p(z) = a_1 + ib_1 + \epsilon(z) \) and \( p(z) \to a_1 + ib_1 \) as \( z \to z_1 \) so \( f \) is complex differentiable at \( z_1 \).  \( \Box \)