Math 336 Sample Problems

One notebook sized page of notes (both sides) will be allowed on the test. The test will be comprehensive. The final exam is at 8:30 am, Monday, June 7, in the regular classroom.

1. Gamelin IX.1, 1,2.

2. Let \( f(z) = \frac{z - a}{1 - \overline{a}z}, \) where \(|a| < 1\). Let \( D = \{z : |z| < 1\} \). Prove that
   \[
   (a) \quad \frac{1}{\pi} \int_{D} |f'(z)|^2 \, dx \, dy = 1.
   \]
   \[
   (b) \quad \frac{1}{\pi} \int_{D} |f'(z)| \, dx \, dy = \frac{1 - |a|^2}{|a|^2} \log \left( \frac{1}{1 - |a|^2} \right).
   \]

3. Let \( u(x, y), \ v(x, y) \) be continuously differentiable as functions of \((x, y)\) in a domain \( \Omega \). Let \( f(z) = u(z) + iv(z) \). Suppose that for every \( z_0 \in \Omega \) there is an \( r_0 \) (depending on \( z_0 \)) such that
   \[
   \int_{|z - z_0| = r} f(z) \, dz = 0,
   \]
   for all \( r \) with \( r < r_0 \). Prove that \( f \) is analytic in \( \Omega \). Hint: Show that \( f \) satisfies the Cauchy-Riemann equations in \( \Omega \).

4. Prove that all conformal maps from the upper half plane to the unit disk have the form
   \[
   \alpha \frac{z - \beta}{z - \overline{\beta}},
   \]
   where \(|\alpha| = 1, \ \text{Im}(\beta) > 0\).
5. Let \( D_2 = \{ z : |z| < 2 \} \) and \( I = \{ x \in \mathbb{R} : -1 \leq x \leq 1 \} \). Find a bounded harmonic function \( u \), defined in \( D_2 - I \) such that \( u \) does not extend to a harmonic function defined in all of \( D_2 \).

6. Find a conformal map from the region between the two lines \( y = x \) and \( y = x + 2 \) to the upper half plane, which sends 0 to 0.

7. Find a function, \( h(x,y) \), harmonic in \( \{ x > 0, y > 0 \} \), such that

\[
h(x,y) = \begin{cases} 
0 & \text{if } 0 < x < 2, y = 0, \\
1 & \text{if } x > 2, y = 0, \\
2 & \text{if } x = 0, y > 0
\end{cases}
\]

8. Suppose that \( u \) is harmonic on all of \( \mathbb{C} \) and \( u \geq 0 \). Prove that \( u \) is constant.

9. Suppose \( f \) is analytic on \( H = \{ z = x + iy : y > 0 \} \) and suppose \( |f(z)| \leq 1 \) on \( H \) and \( f(i) = 0 \). Prove

\[
|f(z)| \leq \left| \frac{z - i}{z + i} \right|.
\]

10. Let \( u(z) \) be harmonic in \( \{ z : 0 < |z| < 1 \} \). Let \( P = \int_{|z|=r} \frac{\partial u}{\partial n} ds \), where \( 0 < r < 1 \). Show that \( P \) does not depend on \( r \). Prove that

\[
u(z) = \frac{P}{2\pi} \log |z| + \text{Re}(f(z))
\]

where \( f \) is analytic in \( \{ z : 0 < |z| < 1 \} \).

11. Prove that if \( |z| < 1 \)

\[
\lim_{n \to \infty} \prod_{k=0}^{k=n} (1 + z^{2k}) = \frac{1}{1 - z}
\]

12. There may be homework problems or example problems from the text on the final. Don’t forget previous sample problems.