Continued Fraction Approximations
of the Riemann Zeta Function

MATH 336

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1 Introduction

Continued fractions serve as a useful tool for approximation and as a field of their own. Here we will concern ourselves with results from Cvijovic and Klinowski from *Continued-Fraction Expansions for the Riemann Zeta Function and Polylogarithms* [3]. From the results, we will be capable of numerically approximating the Riemann zeta function \( \zeta \) for integer values \( n \), which are special cases of the polylogarithm.

2 Notation

We will denote the positive integers \( N \) and \( N \cup \{0\} \) as \( \mathbb{Z}^+ \). We will define the polylogarithm function as follows.

\[
\text{Li}_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu},
\]

In particular, \( \text{Li}_1(1) = \zeta(\nu) \) where \( \zeta(\nu) \) is the Riemann zeta function. We will denote the set of all real-valued, bounded, monotone non-decreasing functions \( \phi(t) \) with infinitely many values on \( a \leq t \leq b \) as \( \Phi(a, b) \) where \( a, b \) are elements of the extended reals \( \mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\} \).

3 Preliminary Definitions and Results

Here we will give necessary definitions and some preliminary results.

3.1 Continued Fractions

We define a continued fraction as follows.

**Definition 3.1.** An (infinite) continued fraction \( K(a_k/b_k) \) is an expression of the form

\[
K(a_k/b_k) = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}},
\]

The \( n \)th approximate \( F_n \) is defined

\[
F_n = \frac{n}{k=1} \frac{a_k}{b_k} = \frac{A_n}{B_n}
\]

We say \( K(a_k/b_k) \) converges to \( F \) if the sequence of approximates converge \( F \) in the extended complex plane \( \mathbb{C}^* = \mathbb{C} \cup \{\infty\} \). We call \( A_n \) the \( n \)th numerator and \( B_n \) the \( n \)th denominator. We say \( K(a_k/b_k) \) diverges if the limit \( \lim_{n \to \infty} F_n \) does not exist. We call each \( a_k \) and \( b_k \) the \( k \)th numerator and denominator, respectively. Note that we will be use the convention that \( a_k \neq 0 \). We say two continued fractions \( K(a_k/b_k) \) and \( K(a_k^*/b_k^*) \) are equivalent, written \( K(a_k/b_k) \cong K(a_k^*/b_k^*) \), if each approximate \( F_n = F_n^* \).
A continued fraction of the form
\[ K(a_k/b_k) = \frac{\infty}{K} \frac{a_k z}{k=1} \] (2)
is called a regular C-fraction (regular corresponding fraction) and a continued fraction of the form
\[ K(a_k/b_k) = \frac{\infty}{K} \frac{a_k}{k=1} \] (3)
is called a modified regular C-fraction. If each \(a_k > 0\), then (2) and (3) are called regular S-fraction and modified regular S-fraction (Stieltjes fractions), respectively.

A finite continued fraction
\[ \frac{\infty}{n} \sum_{k=1} a_k z \] is said to correspond to the series
\[ \sum_{k=0}^{\infty} c_k z^k \] at \(z = \infty\) if the following formal power series expansions are valid:
\[ F_n(z) - \sum_{p=0}^{\lambda_n} c_p z^k = \text{const} z^{-(\lambda_n+1)} + \ldots \]

Where \(n = 1, 2, 3, \ldots\).

### 3.2 The Stieltjes-Riemann Integral

Here we will define the Stieltjes-Riemann integral of a function \(f(x)\), denoted by \(\int_a^b f(x) d\alpha(x)\), and give a few preliminary results. Here, we will use Apostol [1]. We define \(\Delta \alpha_k = \alpha(x_k) - \alpha(x_{k-1})\) such that
\[ \sum_{k=1}^{n} \Delta \alpha_k = \alpha(b) - \alpha(a) \]

We will also use the notion of a partition \(P\) of an interval \([a, b]\). This will be the same as that discussed in Folland [4]. We now define the Stieltjes-Riemann integral.

**Definition 3.2.** Let \(P = \{x_0, x_1, \ldots, x_k\}\) be a partition of \([a, b]\) and let \(t_k \in [x_{k-1}, x_k]\). Then the Stieltjes-Riemann sum of \(f\) with respect to \(\alpha\) is defined as
\[ S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k) \Delta \alpha_k \]

If there exists a unique number \(A\) such that for any \(\epsilon > 0\), there exists a partition \(P_\epsilon\) of \([a, b]\) such that for every partition \(P\) finer than \(P_\epsilon\) and for every choice of \(t_k \in [x_{k-1}, x_k]\), we have that \(|S(P, f, \alpha) - A| < \epsilon\). The number \(A = \int_a^b f(x) d\alpha(x)\).

We state without proof that \(A\) is uniquely determined whenever it exists. For our proof the main theorem, we will need the following two theorems.

**Theorem 3.3.** Suppose \(f\) is continuous on \([a, b]\) and \(\alpha\) is any monotonic, increasing function. Then \(f\) is integrable with respect to \(\alpha\) over \([a, b]\).
For a proof, see [2]. We now give criteria where a Stieltjes-Riemann integral simplifies to a Riemann integral.

**Theorem 3.4.** Suppose $f$ is integrable with respect to $\alpha$ on $[a, b]$. If $\alpha$ is continuously differentiable on $[a, b]$, then $\int_a^b f(x) \alpha'(x) \, dx$ exists. Further

$$\int_a^b f(x) \, d\alpha(x) = \int_a^b f(x) \alpha'(x) \, dx$$

For proof, see Apostol [1].

### 3.3 The Markov Theorem

We will state the Markov theorem, without proof, since it will be used the proof of the main theorem. For a proof, see Perron [6]. However, we will state it as found in Jones and Thron [5].

**Theorem 3.5.** Suppose $\phi \in \Phi(0, a)$. Then there is a modified $S$-fraction which corresponds to the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k \mu_k}{z^k} \quad \text{where} \quad \mu_k = \int_0^a t^k \, d\phi(t) \quad (4)$$

at $z = \infty$, converges to the function

$$\int_0^a \frac{z}{z+t} \, d\phi(t) \quad (5)$$

for all $z \in \mathbb{C} \setminus [-a, 0]$.

### 3.4 Hankel Determinants

**Definition 3.6.** Suppose $\{c_k\}_{k=0}^{\infty}$ is a sequence. Then the Hankel determinants $H^{(r)}_m$ associated with $\{c_k\}$, where $r \in \mathbb{Z}^+$ and $m \in \mathbb{N}$ are given by

$$H^{(r)}_0 = 1, \quad H^{(r)}_m = \begin{vmatrix}
  c_r & c_{r+1} & \cdots & c_{r+m-1} \\
  c_{r+1} & c_{r+2} & \cdots & c_{r+m} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{r+m-1} & c_{r+m} & \cdots & c_{r+2m-2}
\end{vmatrix}$$

### 4 The Main Theorem

**Theorem 4.1.** Suppose that $r \in \mathbb{Z}^+$ is a non-negative integer and $m, n \in \mathbb{N}$. For any fixed $r, m, n$, define $A^{(r)}_m(n)$ as the determinant of an $m \times m$ matrix

$$A^{(r)}_m(n) = \det \left| \frac{(-1)^{i+j+r}}{(r+i+j-1)^n} \right|_{1 \leq i, j \leq m}$$
Where we define $A_0^{(r)}(n) = 1$. Then

$$-Li_n(-z) = \sum_{k=1}^{\infty} \frac{a_{n,k} z}{1}$$

(6)

With

$$a_{n,1} = 1, \quad a_{n,2m} = -\frac{A_0^{(1)}(n) A_{m-1}^{(0)}(n)}{A_0^{(0)}(n) A_{m-1}^{(1)}(n)}, \quad a_{n,2m+1} = -\frac{A_{m-1}^{(1)}(n) A_{m+1}^{(1)}(n)}{A_0^{(0)}(n) A_{m}^{(1)}(n)}$$

(7)

Proof. Consider the function

$$\phi_n(t) = \begin{cases} 
0, & t = 0 \\
\frac{1}{(n-1)!} \int_0^t \left( \log \left( \frac{1}{x} \right) \right)^{n-1} dx, & 0 < t \leq 1 \\
1, & t > 1 
\end{cases}$$

For $n = 1$, the integrand is just 1, so it is clearly integrable and $\phi_n(t)$ is continuous. Where $n \in \mathbb{N}$. Prudnikov [7] gives us

$$\int_{\epsilon}^t \left( \log \left( \frac{1}{x} \right) \right)^{n-1} dx = \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!} \left( t \log t \right)^k - \epsilon \left( \log \epsilon \right)^k$$

(8)

We apply L'Hôpital's rule to get that $\epsilon \log^k \epsilon \to 0$ as $\epsilon \to 0$. So

$$\int_{0}^{t} \left( \log \left( \frac{1}{x} \right) \right)^{n-1} dx = \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!} t \log^k t$$

(9)

L'Hôpital's rule gives that $\phi_n(t) \to 0$ as $t \to 0^+$ and $\phi_n(t) \to 1$ as $t \to 1^-$. For $0 < t \leq 1$, $\log \left( \frac{1}{x} \right) \geq 0$ and continuous and, thus, integrable, so the integral is monotonically increasing and continuous on $[0, 1]$. Further, $\phi_n(t) \in \Phi(0, \infty)$.

Consider the following integral, called the Stieltjes transform of $\phi_n(t)$.

$$f_n(z) = \int_0^{\infty} \frac{d\phi_n(t)}{z + t}$$

(10)

Where $z \notin [-\infty, 0]$. Then by (3.3), the integrand is integrable with respect to $\phi_n(t)$. Further, since $\phi_n(t)$ is continuously differentiable on $[0, \infty)$, by theorem (3.4), we have that

$$f_n(z) = \frac{1}{(n-1)!} \int_0^1 \frac{1}{z + t} \left( \log \left( \frac{1}{t} \right) \right)^{n-1} dt$$

We then substitute $x = \log \left( \frac{1}{t} \right)$. This gives us that $t = e^{-x}$ and $dt = -e^{-x} dx$. So

$$f_n(z) = \frac{1}{(n-1)!} \int_0^{\infty} \frac{x^{n-1}}{z + e^{-x}} (-e^{-x}) dx = \frac{1}{(n-1)!} \int_0^\infty \frac{x^{n-1}}{e^x + \frac{1}{z}} dx$$
This a form of the Fermi-Dirac integral, which has a known polylogarithm representation. In our case

\[ f_n(z) = -Li_n \left( \frac{-1}{z} \right) \]

Using the series representation of the polylogarithm (1), we get, for \(|z| > 1\), the following.

\[ f_n(z) = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k^n z^k} \iff zf_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^n z^k} = \sum_{k=0}^{\infty} \frac{c_{n,k}}{z^k} \]

Where we have let \( c_{n,k} = \frac{(-1)^k}{(k+1)^n} \). Further, Markov tells us there exists a corresponding modified S-fraction that converges to \( zf_n(z) \) for all \( z \in \mathbb{C} \setminus [0,-1] \) and even tells us that

\[ c_{n,k} = (-1)^k \mu_{n,k} \tag{10} \]

Where \( \mu_{n,k} = \frac{1}{(n-1)!} \int_0^1 t^k \left( \log \left( \frac{1}{1+t} \right) \right)^{n-1} dt. \)

Jones and Thron [5] give us that whenever a series \( S = \sum_{k=0}^{\infty} \frac{c_k}{z^k} \) corresponds to a modified C-fraction \( C = \sum_{k=0}^{\infty} \frac{a_k}{1} \) at \( z = \infty \), we know that

\[ a_1 = c_0, \quad a_{2m} = -\frac{H_m^{(1)} H_{m-1}^{(0)}}{H_m^{(0)} H_{m-1}^{(1)}}, \quad a_{2m+1} = -\frac{H_{m-1}^{(1)} H_m^{(0)}}{H_m^{(0)} H_m^{(1)}} \]

Which is exactly what we have, except

\[ a_{n,1} = 1, \quad a_{n,2m} = -\frac{A_m^{(1)}(n) A_{m-1}^{(0)}(n)}{A_m^{(0)}(n) A_{m-1}^{(1)}(n)}, \quad a_{n,2m+1} = -\frac{A_{m-1}^{(1)}(n) A_{m+1}^{(0)}(n)}{A_m^{(0)}(n) A_{m+1}^{(1)}(n)} \]

With each \( A_m^{(c)}(n) \) as described in the main theorem. We then have

\[ zf_n(z) = \frac{a_{n,1}}{1 + \frac{a_{n,2}}{z + \frac{a_{n,3}}{1 + \frac{a_{n,4}}{z + \cdots}}}} \]

Dividing both sides by \( z \) and simple factoring gives us

\[ f_n(z) \cong \frac{a_{n,1}(1/z)}{1 + \frac{a_{n,2}(1/z)}{1 + \frac{a_{n,3}(1/z)}{1 + \frac{a_{n,4}(1/z)}{1 + \cdots}}}} = \sum_{k=1}^{\infty} \frac{a_{n,k}(1/z)}{1} \]
Thus, \(-\Li_n(-1/z) = \lim_{k \to \infty} \frac{a_n,k(1/z)}{1}\). So
\[-\Li_n(-z) = \lim_{k \to \infty} \frac{a_n,k z}{z}\]
And we are done.

5 Additional Results

We conclude with some calculations. Using our results, we may immediately use our results for
\(-\Li_1(-z) = \log(1 + z)\) and \(-\Li_n(-1) = (1 - 2^{1-n})\zeta(n)\), for integers \(n \geq 2\). Cvijović works out the first of these for us.

\[\log(1 + z) = \lim_{k \to \infty} \frac{a_{1,k} z}{1}\]

Where
\[a_{1,1} = 1, \quad a_{1,2m} = \frac{m}{2(2m - 1)}, \quad a_{n,2m+1} = \frac{m}{2(2m + 1)}\]

Take \(z = 1\). Then we should have an approximation for \(\log(2)\). We have
\[\left\{a_{1,k}\right\}_{k=1}^{11} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{10}, \frac{1}{14}, \frac{1}{5}, \frac{3}{18}, \frac{1}{22}\right\}\]

So \(\log(2) \approx 0.69314721238933921\). The more precise value is \(\log(2) \approx 0.6931471805599453\). For \(z = 2\), that is, \(\log(3)\), we multiply each of the \(a_{1,k}\) by 2. This gives us an approximation
\[\log(3) \approx 1.0986122837662749635\] as compared to the more precise \(\log(3) \approx 1.0986122837662749635\).

More appropriately, let \(z = e^{-1}\). Using this value will, of course, give us an exact value for \(\log(e) = 1\) to compare to. We get \(\log(e) \approx 0.69314721238933921\). The exact value of \(\log(e) = 1 + \frac{i\pi}{2}\).

Using Mathematica v.6, we calculate the first 6 numerators of \(a_{n,k}\) for \(1 \leq n \leq 10\) (attached).
With the above table, we calculate the 6th approximants $F_6$ for a given $n$ of the continued fraction expansion of the Riemann zeta function $\zeta(n)$. Below is a table of values for $2 \leq n \leq 10$ accompanied by the values found using Mathematica’s internal command.

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References

1. T. M. Apostol, Mathematical Analysis (2nd Edition), Adison-Wesley, 1974