ON THE GAMMA FUNCTION AND ITS APPLICATIONS

JOEL AZOSE

1. Introduction

The common method for determining the value of $n!$ is naturally recursive, found by multiplying $1 * 2 * 3 * ... * (n-2) * (n-1) * n$, though this is terribly inefficient for large n. So, in the early 18th century, the question was posed: As the definition for the nth triangle number can be explicitly found, is there an explicit way to determine the value of $n!$ which uses elementary algebraic operations? In 1729, Euler proved no such way exists, though he posited an integral formula for $n!$. Later, Legendre would change the notation of Euler's original formula into that of the gamma function that we use today [1].

While the gamma function's original intent was to model and interpolate the factorial function, mathematicians and geometers have discovered and developed many other interesting applications. In this paper, I plan to examine two of those applications. The first involves a formula for the n-dimensional ball with radius r. A consequence of this formula is that it drastically simplifies the discussion of which fits better: the n-ball in the n-cube or the n-cube in the n-ball. The second application is creating the psi and polygamma functions, which will be described in more depth later, and allow for an alternate method of computing infinite sums of rational functions.

Let us begin with a few definitions: The **gamma function** is defined for $\{z \in \mathbb{C}, z \neq 0, -1, -2, \dots\}$ to be:

$$\Gamma(z) = \int_0^\infty s^{z-1} e^{-s} ds \tag{1.1}$$

Remember some important characteristics of the gamma function:

1) For $z \in \{\mathbb{N} \setminus 0\}, \Gamma(z) = z!$

2) $\Gamma(z+1) = z\Gamma(z)$

3) $\ln(\Gamma(z))$ is convex.

The **beta function** is defined for $\{x, y \in \mathbb{C}, Re(x) > 0, Re(y) > 0\}$ to be:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{1.2}$$

Another identity yields:

$$B(x,y) = 2\int_0^{\pi/2} sin^{2x-1}\theta\, cos^{2y-1}\theta d\theta \tag{1.3}$$

Additionally,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{1.4}$$

A thorough proof of this last identity appears in Folland's *Advanced Calculus* [2] on pages 345 and 346. To summarize, the argument relies primarily on manipulation of $\Gamma(x)$ and $\Gamma(y)$ in their integral forms (1.1), converting to polar coordinates, and separating the double integral. This identity will be particularly important in our derivation for the formula for the volume of the n-dimensional ball later in the paper.

With these identities in our toolkit, let us begin.

2. Balls And The Gamma Function

2.1. **Volume Of The N-Dimensional Ball.**

In his article, *The Largest Unit Ball in Any Euclidean Space*, Jeffrey Nunemacher lays down the basis for one interesting application of the gamma function, though he never explicitly uses the gamma function [3]. He first defines the open ball of radius r of dimenision n, $B_n(r)$, to be the set of points such that, for $1 \le j \le n$,

$$\sum x_j^2 < r^2. \tag{2.1}$$

Its volume will be referred to as $\boldsymbol{V_n(r)}$. In an argument that he describes as being "accessible to a multivariable calculus class", Nunemacher uses iterated integrals to derive his formula. He notes that, by definition:

$$V_n(r) = \iint\limits_{B_n(r)} \dots \int 1\, dx_1\, dx_2\, \dots\, dx_n \tag{2.2}$$

By applying (2.1) to the limits of the iterated integral in (2.2) and performing trigonometric substitutions, he gets the following - more relevant - identity, specific to the unit ball, where $r = 1$:

$$V_n = 2V_{n-1} \int_0^{\pi/2} cos^n\theta \, d\theta \tag{2.3}$$

In the rest of *The Largest Unit Ball in Any Euclidean Space*, Nunemacher goes on to determine which unit ball in Euclidean space is the largest. (He ultimately shows that the unit ball of dimension $n = 5$ has the greatest volume, and that the unit ball of dimension $n = 7$ has the greatest surface area, as well as - curiously - noting that V_n goes to 0 as n gets large. While a surprising result, it is not immediately relevant to the topics which I aim to pursue here. If interested, I would refer the reader to Nunemacher's article directly.) Notice, however, that this formula does not use the gamma function. We begin the derivation from here of the Gamma function form.

2.2. **Derivation.** In his 1964 article, *On Round Pegs In Square Holes And Square Pegs In Round Holes* [4], David Singmaster uses the following formula for the volume of an n-dimensional ball:

$$V_n(r) = \frac{\pi^{n/2} r^n}{\Gamma(n/2+1)} \tag{2.4}$$

However, he never shows the derivation of this formula, and other references to Singmaster's article claim that the derivation appears explicitly in Nunemacher's article. I feel this to be an important omission, and I have endeavored here to recreate the derivation for the sake of completeness. We shall begin where Nunemacher left off with equation (2.3).

Recall (1.3) and notice its similarity to (2.3). It quickly becomes apparent that (2.3) may be rewritten as:

$$V_n(1) = V_{n-1}(1)\, B(\frac{1}{2}, \frac{n}{2} + \frac{1}{2}) \tag{2.5}$$

Continuing the recursion, we note:

$$V_{n-1}(1) = V_{n-2}(1)\, B(\frac{1}{2}, \frac{n}{2}) \tag{2.6}$$

Consequently,

$$V_n(1) = V_1(1)\, B(\frac{1}{2}, \frac{3}{2}) \ldots B(\frac{1}{2}, \frac{n}{2})\, B(\frac{1}{2}, \frac{n}{2} + \frac{1}{2}) \tag{2.7}$$

where $V_1(1) = 2 = B(\frac{1}{2}, 1)$. Substituting Gamma for Beta using (1.4) gives:

$$V_n(1) = \left[\frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(\frac{3}{2})} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} \cdots \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)} \right], \tag{2.8}$$

which telescopes to:

$$V_n(1) = \left[\frac{(\Gamma(\frac{1}{2}))^n \Gamma(1)}{\Gamma(n/2+1)} \right] \tag{2.9}$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(1) = 1$,

$$V_n(1) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \tag{2.10}$$

Now the heavy lifting is done. Consider again the recursion relation that we used in (2.3). This recursion relation holds true for the unit ball - that is, when $r = 1$. However, when $r = 1$, we do not see the r in this equation. Instead, when we take the more general form, we get the modified recursion relation:

$$V_n = 2rV_{n-1} \int_0^{\pi/2} cos^n\theta \; d\theta \tag{2.11}$$

Going through the derivation will be virtually identical, except we have dilated the ball's size by a factor of r, and its volume by a factor of r^n. This finally yields:

$$V_n(r) = \frac{\pi^{n/2} r^n}{\Gamma(n/2+1)}, \tag{2.12}$$

which is consistent with with our original statement of (2.4). Now the derivation of the n-ball's volume using the gamma function is complete, and we may proceed to an interesting application.

2.3. **The Packing Problem.** In the motivation for his article, Singmaster explains the purpose of his article: "Some time ago, the following problem occurred to me: which fits better, a round peg in a square hole or a square peg in a round hole? This can easily be solved once one arrives at the following mathematical formulation of the problem. Which is larger: the ratio of the area of a circle to the area of the circumscribed square or the ratio of the area of a square to the area of the circumscribed circle?" [4]

The formula that we derived in the last section will prove invaluable in finding this. Since he is focusing on ratios, Singmaster uses the unit ball in both cases, though it would work similarly with any paired radius.

For the unit ball, the edge of the circumscribed cube is necessarily length 2, since it is equal in length to a diameter of the unit ball. The edge of the n-cube inscribed in the unit n-ball has length $2/\sqrt{n}$, since the diagonal of an n-cube is $\sqrt{n}$ times its edge. (Remember that the diagonal of the n-cube inscribed in the unit n-ball is the diameter of the n-ball.)

So, we construct formulas for the volume of the relevant balls and cubes using (2.4) and the facts which we have just stated:

$$V(n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}, \tag{2.13}$$

$$V_c(n) = 2^n, \tag{2.14}$$

$$V_i(n) = \frac{2^n}{n^{n/2}}, \tag{2.15}$$

where $V(n)$ represents the volume of the unit n-ball (as derived), $V_c(n)$ the volume of the circumscribed cube, and $V_i(n)$ the volume of the inscribed cube. We consider now the ratios of (2.13) to (2.14) - that is, a round peg in a square hole - and that of (2.15) to (2.13) - a square peg in a round hole.

$$R_1(n) = \frac{V(n)}{V_c(n)} = \frac{\pi^{n/2}}{2^n\Gamma\left(\frac{n+2}{2}\right)} \tag{2.16}$$

$$R_2(n) = \frac{V_i(n)}{V(n)} = \frac{2^n\Gamma\left(\frac{n+2}{2}\right)}{n^{n/2}\pi^{n/2}} \tag{2.17}$$

He then takes $\dfrac{R_1(n)}{R_2(n)}$ and applies Stirling's approximation for the gamma function: For z large,

$$\Gamma(z) \sim z^{z-1/2}\, e^{-z} \sqrt{2\pi} \tag{2.18}$$

Singmaster shows that as n goes to infinity, this ratio goes to zero. So, for large enough n, $R_2(n)$ is greater. By simple numerical evaluation, he determines the tipping point to be when $n = 9$. The most important result of this article is the following theorem:

Theorem. *The n-ball fits better in the n-cube better than the n-cube fits in the n-ball if and only if* $n \leq 8$.

3. Psi And Polygamma Functions

In addition to the earlier, more frequently used definitions for the gamma function, Weierstrass proposed the following:

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} (1 + z/n)\, e^{-z/n}, \tag{3.1}$$

where γ is the Euler-Mascheroni constant. Van der Laan and Temme reference another proof of this by Hochstadt [1]. This will be useful in developing the new gamma-related functions in the subsections to follow, as well as important identities. Ultimately, we will provide definitions for the **psi function** - also known as the **digamma function** - as well as the **polygamma functions**. We will then examine how the psi function proves to be useful in the computation of infinite rational sums.

3.1. Definitions.

Traditionally, $\psi(z)$ is defined to be the derivative of $\ln(\Gamma(z))$ with respect to z, also denoted as $\dfrac{\Gamma'(z)}{\Gamma(z)}$. Just as with the gamma function, $\psi(z)$ is defined for $\{z \in \mathbb{C}, z \neq 0, -1, -2, \dots\}$. Van der Laan and Temme provide several very useful definitions for the psi function. The most well-known representation, derived from (3.1) and the definition of $\psi(z)$, is as follows:

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)}, \tag{3.2}$$

though the one that we will ultimately use in the following subsection to compute sums is defined thusly:

$$\psi(z) = -\gamma - \int_0^1 \frac{t^{z-1} - 1}{1 - t}\, dt \tag{3.3}$$

This integral holds true for $Re(z) > -1$, and can be verified by expanding the denominator of the integrand and comparing to (3.2). These two are the most important definitions for the psi function, and they are the two that we will primarily use.

We will now define the **polygamma functions**, $\psi^{(k)}$.This is a family of functions stemming from the gamma and digamma functions. They are useful because they lead to better- and better-converging series. As you might imagine from the notation, the polygamma functions are the higher-order derivatives of $\psi(z)$. Consider these examples from repeated differentiation of (3.2):

$$\psi'(z) = \sum_{n=0}^{\infty} (z+n)^{-2}, \quad \psi^{(k)}(z) = (-1)^{k+1} k! \sum_{n=0}^{\infty} (z+n)^{-k-1} \tag{3.4}$$

Again, we note that, as k increases, $\psi^{(k)}(z)$ becomes more and more convergent. Now, though, we will set aside the polygamma functions and turn our focus back to the psi function and its utility in determining infinite sums.

3.2. **Use In The Computation Of Infinite Sums.** Late in their chapter on some analytical applications of the gamma, digamma, and polygamma functions, van der Laan and Temme state: "An infinite series whose general term is a rational function in the index may always be reduced to a finite series of psi and polygamma functions" [1].

Let us consider the following specific problem to motivate more general results given at the end of this section.

$$\text{Evaluate} \sum_{n=1}^{\infty} \frac{1}{(n+1)(3n+1)}. \tag{3.5}$$

We begin by expressing the summand as u_n, noting that $u_n = \frac{1}{3}\left(\frac{1}{(n+1)(n+1/3)}\right)$.

Then we perform partial fraction decomposition to yield that $\frac{1}{(n+1)(n+1/3)} =$

$\frac{3/2}{n+1} - \frac{3/2}{n+1/3}$, so $u_n = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+1/3}\right)$. Remember the identity that, for all $A > 0$,

$$\frac{1}{A} = \int_0^\infty e^{-Ax} dx \tag{3.6}$$

This identity can be applied, since both denominators of both fractions are necessarily greater than 0. So the sum in (3.5) can be rewritten as:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(n+1)(3n+1)} &= \frac{1}{2}\sum_{n=1}^{\infty}\left[\int_0^\infty e^{-(n+1)x}dx - \int_0^\infty e^{-(n+1/3)x}dx\right] \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\left[\int_0^\infty e^{-nx}e^{-x}dx - \int_0^\infty e^{-nx}e^{-(1/3)x}dx\right] \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\left[\int_0^\infty e^{-nx}(e^{-x} - e^{-(1/3)x})dx\right] \\
&= \frac{1}{2}\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left[\int_0^\infty e^{-nx}(e^{-x} - e^{-(1/3)x})dx\right]\right)
\end{aligned}$$

Remember from the study of infinite series that $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$. When we subtract the first term of the series, $x^0 = 1$, we get the following result:

$$\sum_{n=1}^{N} x^n = \frac{x(1-x^N)}{1-x}. \tag{3.7}$$

Plugging in e^{-x} for x, we see:

$$\sum_{n=1}^{N} e^{-nx} = \frac{e^{-x}(1-e^{-Nx})}{1-e^{-x}} \tag{3.8}$$

Consider the relevant summation. Due to appropriate convergences following from the monotone convergence theorem, we can interchange the summation and integration and continue our manipulations of the sum.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(n+1)(3n+1)} &= \frac{1}{2}\lim_{N\to\infty}\left[\int_0^{\infty}\frac{e^{-x}(1-e^{-Nx})}{1-e^{-x}}(e^{-x}-e^{-(1/3)x})dx\right] \\ &= \frac{1}{2}\int_0^{\infty}\frac{e^{-x}(e^{-x}-e^{-(1/3)x})}{1-e^{-x}}dx\end{aligned}$$

Now we make use of a change of variables. Let $t = e^{-x}$. Consequently, $-e^{-x}dx = dt$. We will make this substitution. The negative sign due to this change of variable cancels with the one created by switching the limits of the integral, to yield the following:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(n+1)(3n+1)} &= \frac{1}{2}\int_0^{1}\frac{t-t^{1/3}}{1-t}dt \\ &= \frac{1}{2}\int_0^{1}\frac{(t-1)-(t^{1/3}-1)}{1-t}dt \\ &= \frac{1}{2}\int_0^{1}\frac{t-1}{1-t}dt-\frac{1}{2}\int_0^{1}\frac{t^{1/3}-1}{1-t}dt\end{aligned}$$

Compare the two integrals on the right hand side of the above equation to the formula for $\psi(z)$ in (3.3). It becomes obvious that the substitution can be made with the psi function to yield our final result:

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(3n+1)} = \frac{1}{2}\psi(4/3)-\frac{1}{2}\psi(2). \tag{3.9}$$

Professor Efthimiou of Tel Aviv University puts forth a theorem regarding series of the form

$$S(a,b) = \sum_{n=1}^{\infty}\frac{1}{(n+a)(n+b)}, \tag{3.10}$$

where $a \neq b$, and $\{a, b \in \mathbb{C}; Re(a), Re(b) > 0\}$ that generalizes the result which we have shown for a specific example above:

Theorem. $S(a,b) = \dfrac{\psi(b+1) - \psi(a+1)}{b-a}$. [5]

Let it be noted that, at present, our utility of psi functions in the calculation of infinite sums is relegated to strictly positive fractions. (Admittedly, even this is handy in a pinch, though it is hardly ideal.) However, I hope that the thorough calculation of this example is proof enough for the reader that this derivation can be made, and that the same argument could be made for a similar - that is, strictly positive - function with a denominator of degree 2. If a doubt persists, I urge the reader to create a rational function of this form and follow the same steps as my proof to derive an equivalence with a sum of psi and/or polygamma functions.

4. Future Works

Van der Laan and Temme propose that *every* infinite series of rational functions may be reduced to a finite series of psi and polygamma functions. This seems plausible, but the statement requires more rigorous examination to be taken as sound. The subjects that I would like to delve the most deeply into are what I touched on at the very end with Prof. Efthimiou's theorem and the limits on the utility of the psi function in the calculation of infinite sums. I think that it would be a worthwhile endeavor to try to formulate an analogue of Efthimiou's theorem for a function with denominator of degree n. Finally, I would like to work on examining what could be done with infinite sums of fractions that are not strictly positive. I would like to determine if there is a similar formula for these series, as well.

5. Conclusion

In the first section of this paper, we provided definitions for the gamma function. We then went through a gamma derivation for the formula of the volume of an n-ball and used that in working with ratios involving inscribed and circumscribed cubes to determine the following:

Theorem. *The n-ball fits better in the n-cube better than the n-cube fits in the n-ball if and only if* $n \leq 8$.

In the second section, we presented the psi - also known as the digamma - function and the family of polygamma functions. We expressed a specific infinite sum as the finite sum of psi functions as motivation for the following more general result:

Theorem. *For* $a \neq b$, *and* $\{a, b \in \mathbb{C}; Re(a), Re(b) > 0\}$, $\sum_{n=1}^{\infty} \dfrac{1}{(n+a)(n+b)} = \dfrac{\psi(b+1) - \psi(a+1)}{b-a}$.

References

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