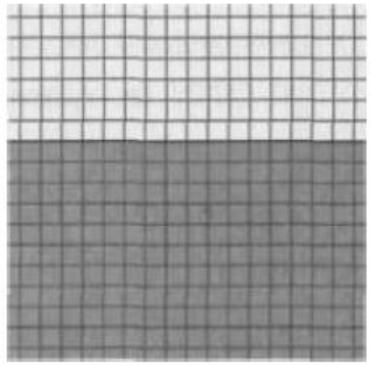
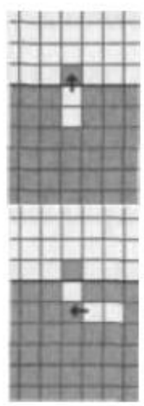
# Conway's Soldiers

Jasper Taylor

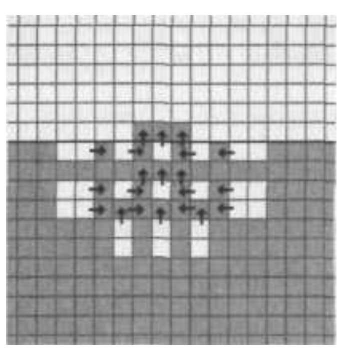
"And the maths problem that I did was called **Conway's Soldiers.** And in **Conway's Soldiers** you have a chessboard that continues infinitely in all directions and every square below a horizontal line has a colored tile on it like this:



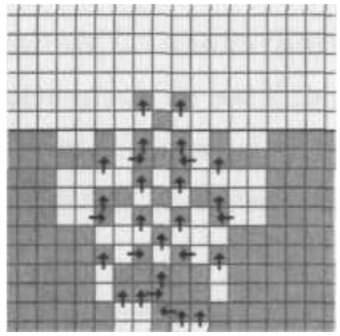
And you can move a colored tile only if it can jump over a colored tile horizontally or vertically (but not diagonally) into an empty square 2 squares away. And when you move a colored tile in this way you have to remove the colored tile that it jumped over, like this:



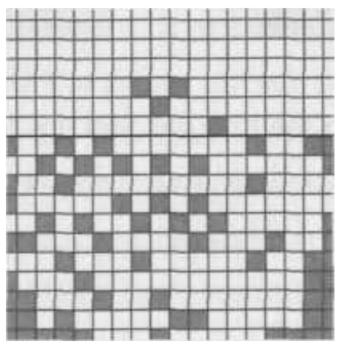
And you have to see how far you get the colored tiles above the starting horizontal line, and you start by doing something like this:



And then you do something like this:



And I know what the answer is because however you move the colored tiles you will never
get a colored tile more than 4 squares above the starting horizontal line, but it is a good maths problem to do in your head when you don't want to think about something else because you can make it as complicated as you need to fill your brain by making the board as big as you want and the moves as complicated as you want. And I got to"



The former is an exact quote from Mark Haddon's fantastic novel **the curious incident of the dog in the night-time,** and is actually how I was introduced to the problem. The purpose of this excerpt is two-fold. First, hopefully it gives the reader a natural feel for the rules of the game that is hard to duplicate with formal mathematical description. Secondly, and more importantly, the autistic hero of this book presents the "maths problem" in a way that is far more intriguing than anything I could manage. When I first read "And I know what the answer is because however you move the colored tiles you will never get a colored tile more than 4 squares above the starting horizontal line" I couldn't believe it and immediately set out to prove him wrong by counter example, it goes without saying that I couldn't and gradually became a believer. Next, I tried my hand at proving it and came up short only to be amazed by how simple, clever and elegant the standard proof is. The subject of this paper is the "maths problem" of **Conway's Soldiers** and its variations.

# Introduction

Conway's Soldiers is based on the game of a peg solitaire which is notoriously infuriating for brute force beginners, while beautifully simple to those who have spent the time to study it. Naturally the puzzle of Conway's Soldiers has very similar qualities.

Conway's Soldiers is a mathematical puzzle popularized and possibly created by James Horton Conway in 1961. The simplest version of the game starts on an infinite checkerboard with pegs, or soldiers, occupying every space on the lower half plane (see figure 1), with pegs that move by jumping over and removing adjacent pegs to the North, South, East, or West. The goal of the game is to advance a peg as far as possible into the upper half plane. There are tons of variations to the game including: different shaped boards, diagonal jumps, n-dimensions, infinite move patterns, and much more. Here is just a small smattering of the possible twists on the classic problem.

# The Basic Game

To begin the analysis of this game we will first look at the simplest version, only allowing vertical and horizontal jumps and using the basic 2-dimensional board. In this version of the game it is impossible to ever advance a soldier five rows above the lower half plane. Remember that when a peg moves it removes the peg it jumps over.

*Proof*

First we will assign every space on the infinite chess board a value and we will call the value of a peg will the value of the square it occupies. Total Peg Value (TPV) will be the sum of all the individual peg values.

Considering the looming infinite sums it would be very convenient if our assigned value for the squares is some sort of power series. Also we would like the value of a peg to increase as it nears the target space in the 5th row. Without loss of generality we will make (0, 5) our target space (using the typical Cartesian coordinates).

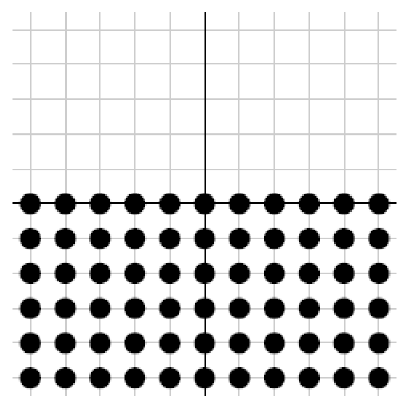


Figure 1: Starting Configuration

So we will give each square the value

$$V(x, y) = a^{|x|+|5-y|} \qquad x, y \in Integers$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $a^4$ | $a^3$ | $a^2$ | $a$ | $a^2$ | $a^3$ | $a^4$ |
| $a^3$ | $a^2$ | $a$ | $T$ | $a$ | $a^2$ | $a^3$ |
| $a^4$ | $a^3$ | $a^2$ | $a$ | $a^2$ | $a^3$ | $a^4$ |
| $a^5$ | $a^4$ | $a^3$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ |
| $a^6$ | $a^5$ | $a^4$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
| $a^7$ | $a^6$ | $a^5$ | $a^4$ | $a^5$ | $a^6$ | $a^7$ |
| $a^8$ | $a^7$ | $a^6$ | $a^5$ | $a^6$ | $a^7$ | $a^8$ |
| $a^9$ | $a^8$ | $a^7$ | $a^6$ | $a^7$ | $a^8$ | $a^9$ |

Figure 2: Values of spaces where T is the target space, and red font denotes a space that initially contains a peg

It will be convenient to think in terms of distance along the lattice from the target space instead of in terms of x and y so we will let

$$d = |x| + |5 - y| \quad \text{And} \quad V(d) = a^d$$

Next we must choose a value of $a$ such that no move will increase the TPV. There are three types of moves.

(i) Moves toward the target space --- $a^d + a^{d-1} \to a^{d-2}$

(ii) Moves away from the target space --- $a^d + a^{d+1} \to a^{d+2}$

(iii) Moves jumping over the $x = 0$ line --- $a^d + a^{d-1} \to a^d$

Using the notation: $\textit{Jumping Piece's Starting Value} + \textit{Jumped Over Piece's Value} \to \textit{Jumping Piece's Ending Value}$

From this we will define Move Value (MV) in the obvious manner

$$MV = \textit{Jumping Piece's Ending Value} - (\textit{Jumping Piece's Starting Value} + \textit{Jumped Over Piece's Value})$$

Intuitively speaking we would expect moves of type (i) to be the best so we will set their move value to zero to find an appropriate value of $a$.

$a^{d-2} - a^{d-1} - a^d = 0$ Multiplying by $a^2$ and factoring out $a^d$ we have: $a^d(-a^2 - a + 1) = 0$

Whose solutions are: $a = \varphi$ and $a = \varphi - 1$ where $\varphi$ is the golden ratio. Because we are going to want power series to converge we will choose $a = \varphi - 1$, but for most of our manipulations we will just be able to use the property $a^2 = 1 - a$. But first we must verify that moves of type (ii) and type (iii) both have $MV < 0$.

For type (ii) moves we have: $MV = a^{d+2} - a^{d+1} - a^d$ since $a < 1$, $a^{d+2} < a^{d+1} < a^d$

$$MV = a^{d+2} - a^{d+1} - a^d < 0$$

And for type (iii) moves we have: $MV = a^d - a^{d-1} - a^d = -a^{d-1} < 0$

Next we compute the TPV of the initial condition

$$TPV = \underbrace{\sum_{n=0}^{\infty} a^5 \cdot a^n}_{\text{From x=0 column}} + \underbrace{2\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} a^5 \cdot a^j \cdot a^k}_{\text{From |y|= j columns}}$$

Applying the formula for summing an infinite series $\sum_{n=0}^{\infty} Cx^n = \frac{C}{1-x}$ to our equation for TPV

$TPV = \frac{a^5}{1-a} + 2\sum_{j=1}^{\infty} \frac{a^5 \cdot a^j}{1-a}$ Recalling that $a^2 = 1 - a$ And re-indexing $j$ to start at 0 we get

$$TPV = a^3 + 2\sum_{j=0}^{\infty} a^4 \cdot a^j = a^3 + \frac{2a^4}{1-a} = a^3 + 2a^2 = a^2(a+2) = (1-a)(a+2)$$
$$= -a^2 - a + 2 = -(1-a) - a + 2 = 1$$

On the other hand a single peg at our target square has a value of $a^0 = 1.$ So, we can only reach our target space if we use every peg in the half infinite plane clearly it is impossible to move every peg so we can never reach our target square.

# The N-Dimensional Game

Next we will consider the n-dimensional game. In this game we will still only allow "straight jumps" meaning that a legal move will only affect pegs in one of the dimensions. Let's see if there is any limit to how far a peg can advance in the n-dimensional game. First let's consider a one dimensional game, this game is quite boring as there is only one possible first move, jumping over the first peg with the second peg in the line, and it is quite obvious that a peg will never be able to reach the 2 rows past the initial setup. So using the one and two dimensional games as evidence, and extrapolating linearly we make the following conjecture:

$R(n) \leq 3n - 2$ where $R(n)$ is the maximum number of rows above the lower half space that a peg can possibly advance to and $n$ is the number of dimensions.

_Proof_

Define the value of each square as $V(x_1, x_2, \ldots, x_n) = a^{R(n)} \cdot a^{|x_1|+|x_2|+\cdots+|x_n|}$ where $a = 1 - \varphi$. Since there are still only the same three move types this guarantees that no legal move will increase the TPV. Recall that $a$ has the following properties: $a + 1 = \frac{1}{a}$ and $1 - a = a^2$.

Let us first consider a function $T(n) = \sum_{x_1, x_2, \ldots, x_n \in Integers} a^{|x_1|+|x_2|+\cdots+|x_n|}$

So $T(n)$ is the sum of $a^d$ over an entire n dimensional space where d is the distance travelled along the lattice from the origin.

$$T(n) = \sum_{x_1, x_2, \ldots, x_n \in Integers} a^{|x_1|+|x_2|+\cdots+|x_n|} = \sum_{-\infty}^{\infty} a^{|x_n|} \sum_{x_1, x_2, \ldots, x_{n-1} \in Integers} a^{|x_1|+|x_2|+\cdots+|x_{n-1}|}$$

$$= T(n-1) \sum_{-\infty}^{\infty} a^{|x_n|} = T(n-1) * \left(1 + \frac{2a}{1-a}\right) = T(n-1)\ \frac{1+a}{1-a} = T(n-1)\frac{1/a}{a^2}$$

$$T(n) = T(n-1)\frac{1}{a^3}$$

It is easy to show that $T(1) = \frac{1}{a^3}$ so by converting the recursive relationship we find that $T(n) = \frac{1}{a^{3n}}$

Now we compute the TPV of the initial condition. The initial condition of the n-dimensional game has pegs placed in every space in the n-dimensional lower half space. We can think of the lower half space as an infinite layering of $n-1$ dimensional hyperplanes the closest being a distance $R(n)$ away.

$$TPV = \sum_{n=0}^{\infty} a^{R(n)} \cdot a^n \cdot T(n-1) = \frac{a^{R(n)}}{a^{3(n-1)}(1-a)} = \frac{a^{R(n)}}{a^{3(n-1)}a^2} = a^{R(n)-3n+3-2} = a^{R(n)-3n+1}$$

So if $R(n) = 3n-1$, $TPV(initial\ condition) = 1$ which is the same TPV as a single peg in the target space. So we would have to use every single one of the infinite number of pegs to reach our target space so $R(n) < 3n-1$ hence $R(n) \leq 3n-2$.

# Reaching the 5th Row

We now return our focus to the basic game. We proved earlier that in the normal two dimensional version of Conway's Soldiers that it is impossible to advance a soldier to the 5th row using a finite number of pieces, however there still might be a way to reach the 5th row with an infinite number of pieces. In order to do this we are going to have to accept some possibly disturbing thoughts. The main issue is we will want to be able to talk about doing an infinite number of moves, letting the process complete and then doing infinitely more moves. In order to do this we will introduce a new parameter $t$ which it is convenient to think of as time. This way we can discretely map an infinite number of moves into a limited amount of time. We will also then use the idea of time to introduce the idea of the un-move

Let's start by considering an infinite set of upward jumps along a single column of moves (See Fig 3). Starting at $t = 0$ and making move $M$ at time $t = \frac{M}{M+1}$ Notice time is always increasing and bounded from above by $t = 1$ so that all of the moves happen in the time interval $0 \leq t \leq 1$.

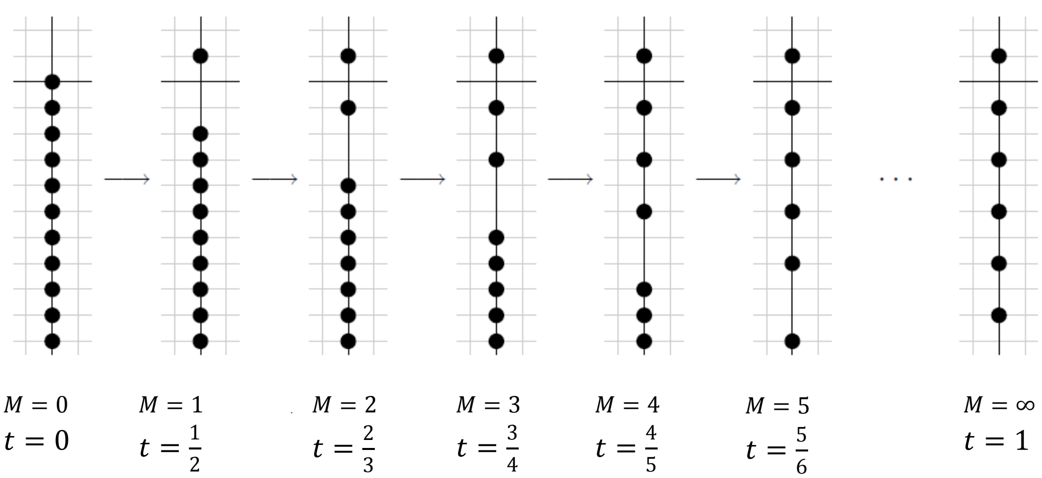



Figure: 3

We will now address the idea of an un-move. With an un-move the idea is to work backwards, where could two pegs have been to make a single peg in this space. When pegs move they remove pegs; when pegs un-move they leave new pegs in their wake (see Fig 4). Let's now consider an infinite series of downwards un-moves starting with just a single peg (see Fig 5). This time since we are using un-moves we need time to decrease with every un-move so we will start at $t = 2$ and make each un-move $U$ at $t = \frac{U+2}{U+1}$ this way we can map the infinite set of un-moves to the time interval $2 \geq t \geq 1$.

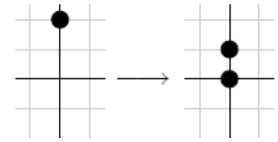


Figure 4: An example of an un-move

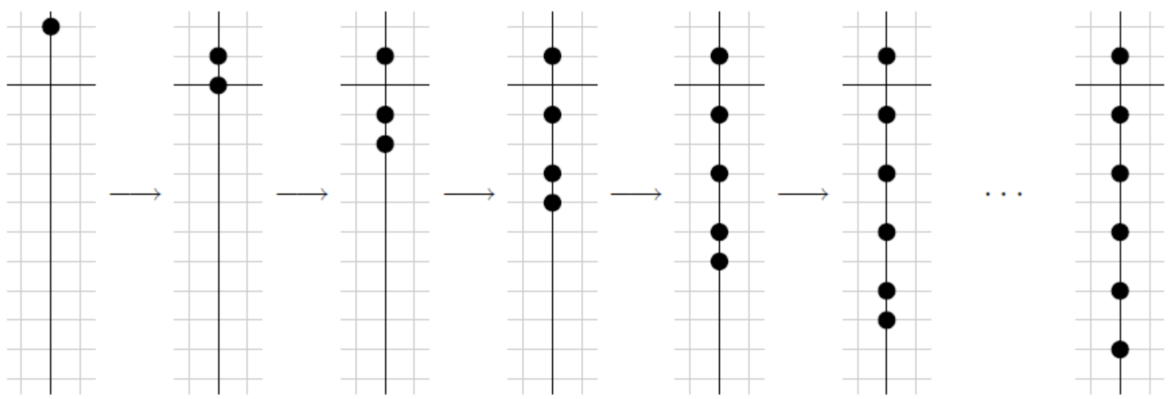


Figure 5

| $U = 0$ | $U = 1$ | $U = 2$ | $U = 3$ | $U = 4$ | $U = 5$ | $U = \infty$ |
|---|---|---|---|---|---|---|
| $t = 2$ | $t = \frac{3}{2}$ | $t = \frac{4}{3}$ | $t = \frac{5}{4}$ | $t = \frac{6}{5}$ | $t = \frac{7}{6}$ | $t = 1$ |

Aha! That is the same configuration that we had after our infinite series of forward moves, so if we start with a half infinite line of pegs at at $t = 0$ and continue until $t = 2$. First we do our series of moves to change our half infinite line of pegs into a half infinite line of alternating empty and filled spaces and then we do our un-moves in reverse to change the alternating half infinite line into a single peg. The next question is where does this peg land with respect to the original half infinite line of pegs. Note that it is very simple to compress the whoosh into compact time intervals, and it will therefore also be easy to compress multiple whooshes into a compact time interval. From this point forward we will ignore this procedure as it is easy, tedious, and boring.

<u>*Claim*</u> The single peg will land 2 spaces beyond the half infinite line.

Assign each peg a value of $V = a^d$ where $a = 1 - \varphi$ and $d$ the distance from our target space 2 spaces beyond the half infinite line. First note that all of our moves were upwards and all of our un-moves were downwards so when we run the clock forwards all of our moves are upwards. These are all moves of type (i) so the TVP of our final peg must be the same as the TVP of the initial half line. The TVP of the initial half infinite line is given by

$TVP = \sum_{n=0}^{\infty} a^2 \cdot a^n = \frac{a^2}{1-a} = 1$ so our final peg must be located in a space where $a^d = 1$ so $d = 0$ and the only space that is 0 distance from our target space is the target space itself. So we now have a sequence of moves that turns a half infinite line of pegs into a single peg 2 rows past the initial half infinite line. We will call this sequence of moves a whoosh (Term borrowed from Tatham and Taylor) (see Fig 5).

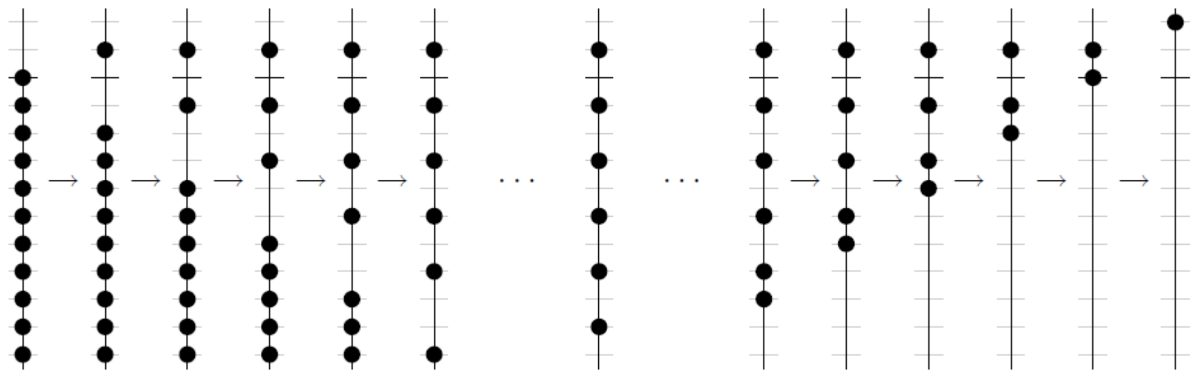
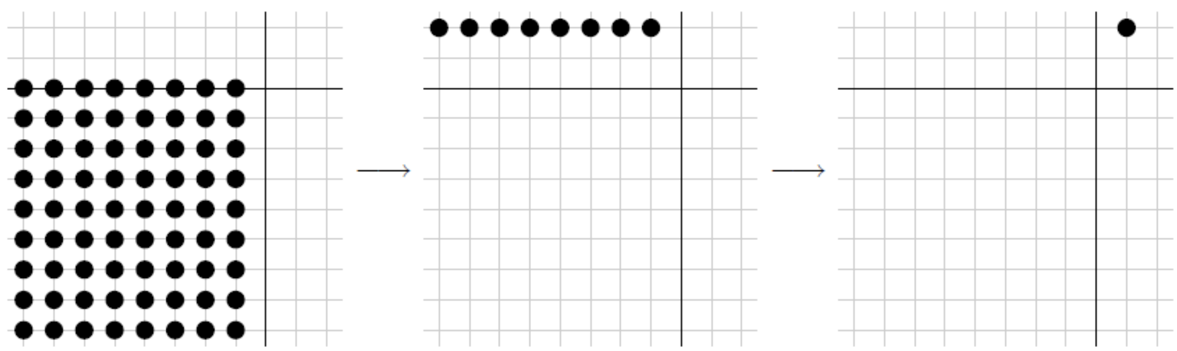
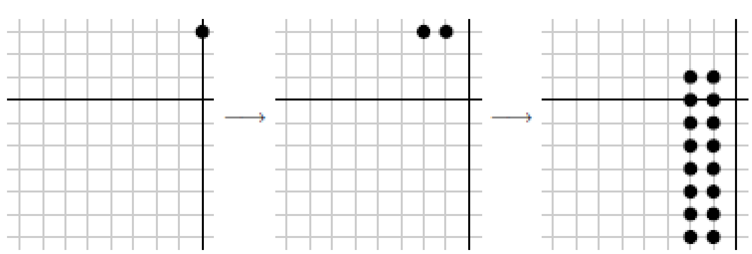


Figure 5: "Whoosh"

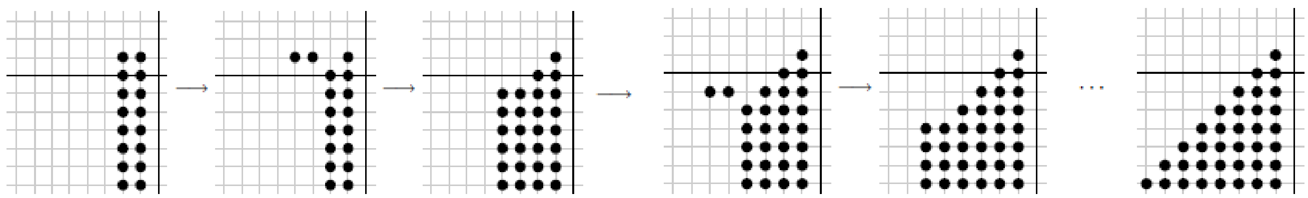
Next we must construct a series of whooshes that whooshes our infinite half plane of pegs into a single peg. We will start by concentrating only on an infinite quarter plane lets use the third quadrant in other words $x, y < 0$ we could simply whoosh each column upwards and then whoosh the $y = 2$ row finishing with a single peg at $(1,2)$. Unfortunately this method is not the solution we want because if we wish to reach our target space $(0,5)$ we can never make anything but moves of type (i) and the final move of the proposed sequence crosses the $x = 0$ line and is therefore of type (iii).



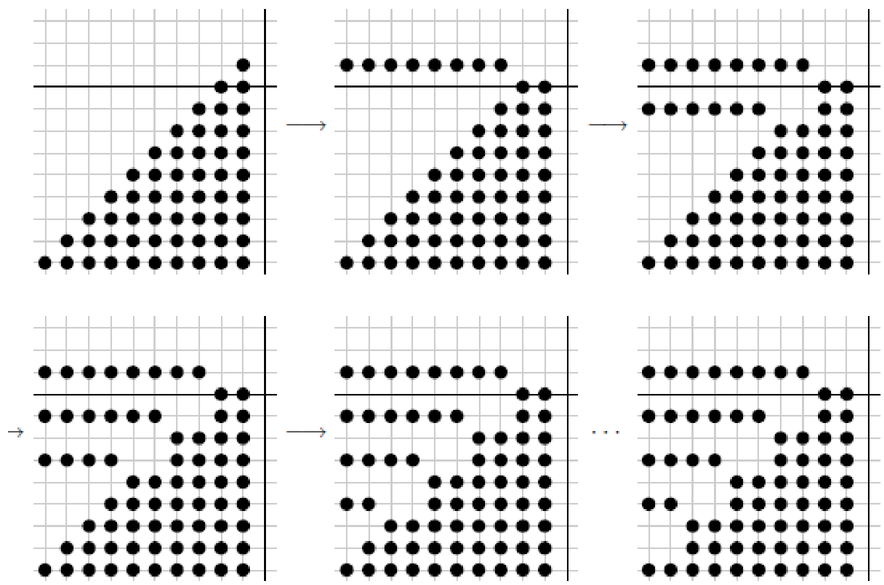
So we must find another way. This time we will work backwards starting with a peg at $(0,3)$ we un-move it to the left resulting in pegs at $(-1,3)$ and $(-2,3)$. And then un-whoosh both pegs downwards.



Now we repeat those same steps always starting with the uppermost peg in the left most column.

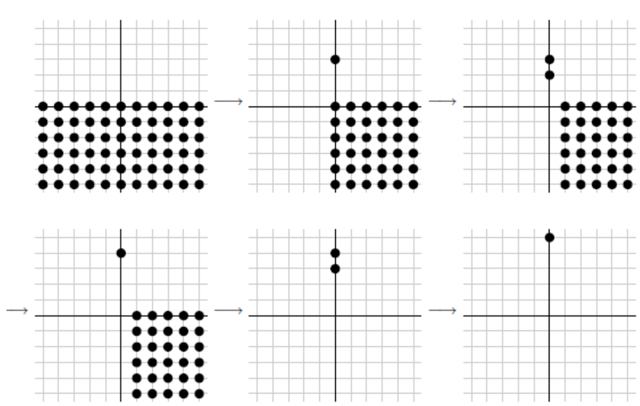


Next we un-whoosh (to the left) every other peg on the left of the staircase starting with the peg at $(-1,1)$ .



Leaving only a finite number of downwards un-moves in each of the columns. Performing those un-moves we are left with a quarter plane of pegs in the spaces $x, y < 0$. We will call this a "mega-whoosh".

Now, we have all the tools to reach the fifth level, first we take the quarter plane $x, y < 0$ and mega-whoosh it to (0, 3). Then we take the $x = 0$ and whoosh it to (0, 2). Next we take the peg at (0, 2) and jump the peg at (0, 3) leaving us with a single peg at (0, 4). Finally we mega-whoosh the quarter plane $x, y > 0$ to (0,3) and use it to jump over the peg at (0,4) leaving us with just one peg at (0,5). Or in pictures …



# Reaching Row $3n-1$

Earlier we proved that in the n-dimensional variation a peg could never advance to the row $3n-1$. However, just like the 2 dimensional case, if we allow infinite moves we can reach one row further. We will prove this by induction.

*Proof*:

We have already shown the base case ($n=1$, and $n=2$ for that matter) so we now assume that it is possible to reach row $3n-1$ in the n-dimensional variation, and now must show that you can reach row $3(n+1)-1$ in the $(n+1)$ dimensional case.

Before we begin we should once again be cognizant of the fact that the initial position in the $(n+1)$ dimensional case is an infinite layering of n-dimensional initial conditions. For convenience sake let's name the dimensions. Specifically let's name them $(x_1, x_2, x_3, \ldots, x_n, x_{n+1})$. Next we will consider the set of 2-dimensional planes defined by $x_3, x_4, \ldots x_n, x_{n+1} = constant$. We have already proved the theory for 1 and 2 dimensions so there is no problem in assuming that we have at least 3 dimensions. On each of these planes we will consider the quarter plane $x_1, x_2 < 0$. On each of these quarter planes we will perform a "near-mega-whoosh."

Starting with a quarter plane, we make the full exact same series of moves, but this time we stop just before we are done, leaving us with a two n-dimensional half hyper planes defined by $x_1 = -1,\ x_2 \leq 1$ and $x_1 = -2,\ x_2 \leq 1$. The step we stop at is illustrated in figure 6.

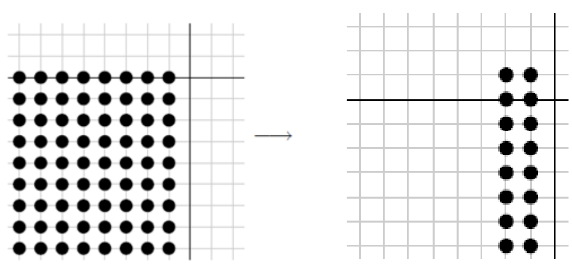


Figure 6

Now we use our assumption that an n-dimensional half hyper plane can reach a space $3n-1$ rows away. So these two half infinite hyper planes can be whooshed into two pegs. One at $(1, 3n-1+1,\ 0, \cdots 0)$ and the other at $(2, 3n-1+1,\ 0, \cdots 0)$. Jumping the latter over the former we get a single peg at $(0, 3n,\ 0, \cdots 0)$, we will call this a "super-mega-whoosh". Now we can simply super-mega-whoosh the quarter plane $x_1, x_2 < 0$ to $(0, 3n,\ 0, \cdots 0)$. Then mega-whoosh the set of pegs satisfying $x_1 = 0$ to $(0, 3n-1,\ 0, \cdots 0)$. Next jump the $(0, 3n-1,\ 0, \cdots 0)$ peg over the $(0, 3n,\ 0, \cdots 0)$ peg landing at the space $(0, 3n+1,\ 0, \cdots 0)$. Now super-mega-whoosh the quarter plane $x_1, x_2 > 0$ to $(0, 3n,\ 0, \cdots 0)$ which we use to jump over the peg at $(0, 3n+1,\ 0, \cdots 0)$ landing in the space $(0, 3n+2,\ 0, \cdots 0)$. Which is obviously in a space $3(n+1)-1$ squares away from the origin. Thus, using infinitely many moves and $n$ dimensions worth of pieces it is possible to reach row $3n-1$.

# Conclusion

These are just a few of the variations of Conway's soldiers and each one seems to have its own beautiful logic to it. Some future puzzles for an interested reader to consider are: allowing diagonal jumps, putting a lower bound on the n-dimensional case, different shaped boards and computing the minimum number of pieces it takes to reach a given row.

# One Final Note

This puzzle is just another example of how mathematics tends to be unreasonably connected. The fact that this idea for a proof actually works speaks tremendously to the power and beauty of mathematics. Who would have ever guessed that $\varphi$ is as deeply ingrained in the game of peg solitaire as $\pi$ is rooted in a circle. Absolutely Marvelous!

## Works Cited

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