Betting with the Kelly Criterion

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1 Introduction

Gambling in all forms, whether it be in blackjack, sports, or the stock market, must begin with a bet. In this paper, we summarize Kelly's criterion for determining the fraction of capital to wager in a gamble. We also test Kelly's criterion by running simulations.

In his original paper, Kelly proposed a different criterion for gamblers. The classic gambler thought to maximize expected value of wealth, which meant she would need to invest 100% of her capital for every bet. Rather than maximizing expected value of capital, Kelly maximized the expected value of the utility function. Utility functions are used by economists to value money and are increasing as a function of wealth under the assumption that more money can never be worse than less [1]. Kelly took the base 2 logarithm of capital as his utility function [2], but we will use the base e logarithm (the natural log) instead.

2 Kelly Criterion

The following derivation is modified from Thorp [1]. We assume that the probability of events are known and independent and that the probability of a win is p $(1 > p > 1/2)$ and the probability of a loss is $q = 1 - p$. Suppose a fraction f $(0 < f < 1)$ of the capital is bet each turn and W_n and L_n represent the number of wins and losses after n bets, respectfully. Rather than even payoff (i.e., a win of 1 unit per unit bet per win), we consider the more general scenario that b units are won per unit bet per win and a units are lost per unit bet per loss. Given initial capital X_0, the capital after n bets is

$$X_n = X_0(1 - af)^{L_n}(1 + bf)^{W_n}.$$

Now define

$$g(f) = log\left(\frac{X_n}{X_0}\right)^{\frac{1}{n}} = \frac{1}{n}\left(L_n \cdot log(1 - af) + W_n \cdot log(1 + bf)\right),$$

the exponential rate of increase per trial. The expected value of $g(f)$ is

$$G(f) = E(g(f)) = q \cdot log(1 - af) + p \cdot log(1 + bf)$$

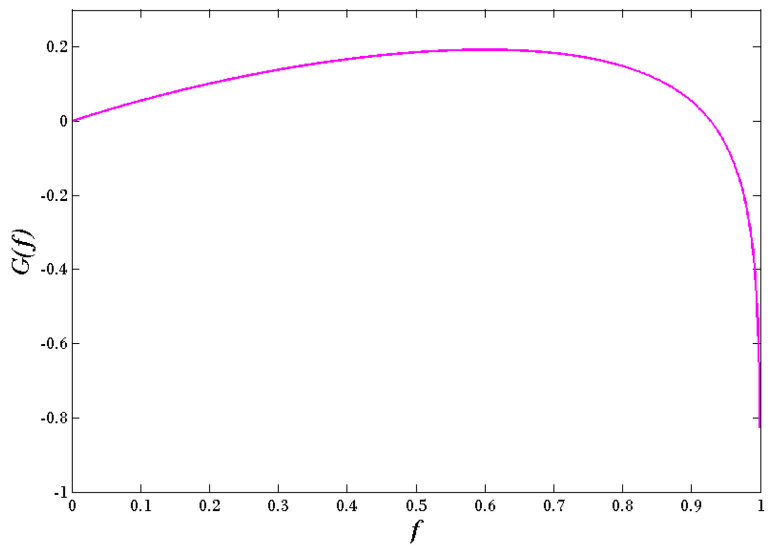
because the ratio of expected wins or losses to trials is given by the probabilities p and q, respectively. We want to maximize $G(f)$ because

$$G(f) = E(g(f)) = E\left(\frac{1}{n} \cdot log(X_n) - \frac{1}{n} \cdot log(X_0)\right),$$

so maximizing $G(f)$ would in turn maximize $E(log(X_n))$, the expected value of the logarithm of wealth. A critical point of $G(f)$ can be found by setting the derivative to 0:

$$G'(f) = -\frac{aq}{1 - af} + \frac{bp}{1 + bf} = 0$$

Figure 1: Expected Value of Logarithm of Wealth vs. Bet as a Fraction of Wealth



$$= \frac{bp - aq - abf}{(1 + bf)(1 - af)} = 0,$$

so the critical point is at

$$f = f^* = \frac{bp - aq}{ab}.$$

Notice that $f^* \neq \frac{1}{a}$, so $G'(f)$ is defined at f^*, and the critical point is a zero of $G'(f)$ there. Since

$$G''(f) = -\frac{a^2 q}{(1 - af)^2} - \frac{b^2 p}{(1 + bf)^2} < 0,$$

f^* is a local maximum. And because $G(0) = 0$ and $\lim_{f \to 1^-} G(f) = -\infty$, the maximum of $G(f)$ is at f^*. Figure 1 shows the plot of $G(f)$ as a function of f. For this function, we set $p = 0.8$, $q = 0.2$, and $a = b = 1$. The maximum occurs at $f^* = p - q = 0.6$.

3 The Stock Market

Kelly criterion can be applied to the stock market. In the stock market, money is invested in securities that have high expected return [3]. The following derivation is modified from Thorp [1]. Since there is not a finite number of outcomes of a bet on a security, we must use continuous probability distributions. Let X

be a random variable that denotes the return per unit, and suppose

$$P(X = \mu + \sigma) = P(X = \mu - \sigma) = \frac{1}{2}.$$

Then the expected value E(X)=μ, and the variance of X is σ^2 (with standard deviation σ). Suppose the initial capital is Y_0 and the bet as a fraction of wealth is f. Then the capital $Y(f)$ is given by

$$Y(f) = Y_0(1 + (1-f)r + fX),$$

where r is the rate of return of capital invested elsewhere. Using the probability assumptions, this means

$$G(f) = E\left(log\left(\frac{Y(f)}{Y_0}\right)\right)$$

$$= \frac{1}{2}log\left(1 + (1-f)r + f(\mu+\sigma)\right) + \frac{1}{2}log\left(1 + (1-f)r + f(\mu-\sigma)\right).$$

If there are n time steps of equal length in the time interval, then we have X at each of those steps, X_i, with i=1, 2,..., n. Also,

$$P\left(X_i = \frac{\mu}{n} + \sigma n^{-\frac{1}{2}}\right) = P\left(X_i = \frac{\mu}{n} - \sigma n^{-\frac{1}{2}}\right) = \frac{1}{2}$$

given that we want the same total μ, σ^2 and r. Then we have

$$\frac{Y_n(f)}{Y_0} = \prod_{i=1}^{n}\left(1 + (1-f)\frac{r}{n} + fX_i\right)$$

and

$$G_n(f) = E\left(log\left(\frac{Y_n(f)}{Y_0}\right)\right) = E\left(\sum_{i=1}^{n} log\left(1 + (1-f)\frac{r}{n} + fX_i\right)\right)$$

$$= \sum_{i=1}^{n}\frac{1}{2}log\left(1 + (1-f)\frac{r}{n} + f\left[\frac{\mu}{n} + \sigma n^{-\frac{1}{2}}\right]\right) + \frac{1}{2}log\left(1 + (1-f)\frac{r}{n} + f\left[\frac{\mu}{n} - \sigma n^{-\frac{1}{2}}\right]\right)$$

$$= \frac{n}{2}log\left(\left(1 + (1-f)\frac{r}{n} + f\frac{\mu}{n}\right)^2 - f^2\sigma^2 n^{-1}\right).$$

Now we expand $G_n(f)$ as a Taylor series around $f = 0$. Calculating the derivatives of $G_n(f)$, we get

$$G_n(0) = n \cdot \frac{r}{n} + O(n^{-\frac{1}{2}})$$

$$\frac{dG_n(0)}{df} = n \cdot \frac{\mu - r}{n} + O(n^{-\frac{1}{2}})$$

$$\frac{d^2G_n(0)}{df^2} = n \cdot -\frac{\sigma^2}{2n} + O(n^{-\frac{1}{2}})$$

and

$$\frac{d^k G_n(0)}{df^k} = O(n^{-\frac{1}{2}})$$

for k $\geq$ 3. Then G_n can be expressed as

$$G_n(f) = r + (\mu - r)f - \sigma^2 \frac{f^2}{2} + O(n^{-\frac{1}{2}}).$$

To make this continuous, we allow $n \to \infty$; thus G_n becomes

$$G_\infty(f) = r + (\mu - r)f - \sigma^2 \frac{f^2}{2}.$$

Notice that $f < 0$ is allowed and is equivalent to taking a short position. This G_∞ is an instantaneous growth rate, so adjustments must be made when Y_n undergoes a change. Using the method in section 2, we find that the optimal betting fraction, f^*, is

$$f^* = \frac{\mu - r}{\sigma^2}.$$

4 Simulations

Using MATLAB, we simulated betting with two different strategies: one using the Kelly Criterion and another with constant betting. The scenario is smplified such that the probability of a win and a loss are known and constant. This may be realistic in the case of a very consistent sports team for example. The parameters given are

probability of winning the bet $p = 0.55$,

probability of losing the bet $1 - p = q = 0.45$,

units won per unit bet per win $b = 10/11$,

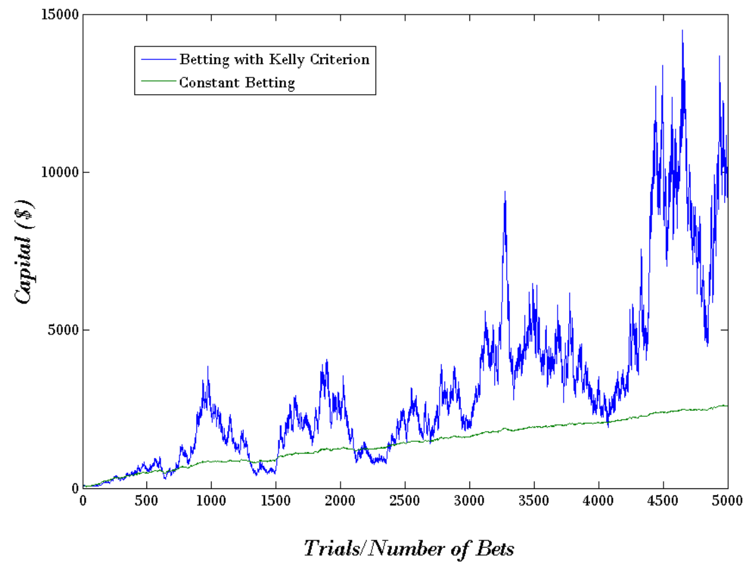
units lost per unit bet per loss $a = 1$,

number of trials $n = 5000$,

and initial capital $X_0 = \$100$.

The rand command in MATLAB was used to generate random numbers for determining the outcome of each trial; this command returns pseudorandom numbers from a uniform distribution. The results are shown in Figure 2.

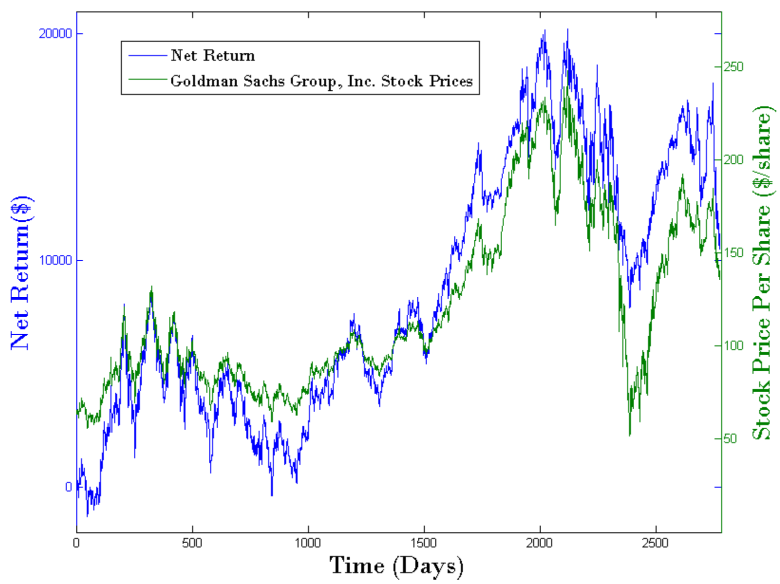
Figure 2: Capital Through 5000 Bets: Betting with the Kelly Criterion vs. with constant bets.



From the graph, betting with the Kelly Criterion clearly has an advantage over constant betting. After 5000 bets, betting with the Kelly Criterion yields a total capital of between \$5000 and \$10000 (a percent increase of capital of over 4900%) while constant betting yields a total capital of around \$2500 (a percent increase of capital of about 2400%). However, unlike the Kelly Criterion curve, constant betting showed a roughly linear trend line; the fluctuations cannot be measured readily by glance. With the Kelly Criterion, the fluctuation is orders of magnitude different though the overall upward trend is above that of constant betting. Noticeable drops and gains of thousands of dollars within 100 bets are evident from looking at the Kelly Criterion graph. In addition, betting with the Kelly Criterion may occasionally be worse than constant betting even after several thousand bets.

The number of bets considered here should also be discussed. Betting 5000 times may be unrealistic for most. If 3 bets were made every week, it would take around 32 years to reach 5000. During this time, even a consistent team would likely not carry the same win percentage! For the short term, it may be better to look at the performance of betting with the Kelly Criterion through 150 bets (1 year's worth of betting). In this interval, the Kelly Criterion seems virtually identical to constant betting. There does not seem to be a significant increase in capital during that time with either method. Appreciable differences are seen only at around 1000 bets, so in order to experience the advantage of using the Kelly Criterion, a bettor should start with more capital, make more bets, or be

Figure 3: Net Return Through 10 Years: Investing in Goldman Sachs using the Kelly Criterion.



willing to wait a long time. From this simulation, we see that betting with the Kelly Criterion is effective after many trials but also quite volatile.

Use of the Kelly Criterion is further investigated through application to the stock market. The closing stock prices of Goldman Sachs Group, Inc. (GS) from May 30, 1999 to May 24, 2010 were obtained [4] and used as the data. In this period, the stock rose from 64.19 to 136.69. Since stocks typically experience many highs and lows, one single mean and standard deviation value cannot represent the behavior of the stock through 11 years accurately. Thus, the data was split into nineteen 146 day blocks, and the mean and standard deviation of each block was found. The optimal fraction for each block could then be calculated. The parameters given are

$$\text{return rate of other investments } r = 0.00,$$

$$\text{number of days} = 2774,$$

$$\text{initial investment } Y_0 = \$10000.$$

The stock price is the price per share, so the number of shares for day k was given by the investment for that day as suggested by the Kelly Criterion divided by that day's stock price. Fractional shares were allowed. The subsequent value of those shares was the product of the number of shares for day k and the stock price on day $k + 1$. To simplify matters, the rate of return of the uninvested wealth was set to zero. Hence the total wealth was the sum of the uninvested wealth and the value of the invested wealth. This yielded the net return when subtracted by the initial investment. It should be noted that the fraction of wealth to invest was limited to ≤ 1 so that we did not have to deal with short selling or debt. The results can be seen in Figure 4.

The results are similar to those found in the case of sport betting. The net return after 11 years is about \$10000, which is 100% of the initial investment. While investing higher fractions of wealth would increase the net return slightly, that is an extremely risky strategy when the future stock price is unknown. The Kelly Criterion clearly involves nontrivial risk, as evidenced by the negative return within the first 100 days; however, the risk is reduced by the changing of fraction of wealth invested.

This was a simplified example, so the actual outcome would differ if, e.g., the uninvested wealth were put into a risk free security, or if short selling or debt were considered so that the fraction of wealth invested could be above 1. Still, this simulation provides insight into how the Kelly Criterion might perform when used on the stock market.

5 Conclusion

The Kelly Criterion can be utilized to find the optimal bet size for a wager. Not only can Kelly Criterion be used for sports betting and casino games, it can also be used in the stock market. We derived the optimal bet size expression for a situation with only two outcomes and discrete time steps. Furthermore, we used continuous probability distributions to find the optimal bet size expression in a situation where securities may be bought or sold. Finally, we simulated a betting situation using MATLAB and compared the results of betting with the Kelly Criterion to constant betting. This was expanded to investing in the stock market. We found that the Kelly Criterion is effective, but initial capital should be high and/or a great deal of time should be allowed for the final capital to reach substantial amounts. In this way, the Kelly Criterion is impractical and so is not applied in many situations.

References

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