Prime Number Theorem

This is an outline of the Nevanlinna-Zagier-Garabini proof of the Prime Number Theorem (PNT)

Notation

\[ \pi(x) = \# \text{primes } \leq x, \quad P = \text{set of primes} \]

\[ \mu(x) = \sum_{\substack{p \leq x \\ p \in P}} \log p \]

\[ \phi(s) = \sum_{\substack{p \in P \\ p^s}} \frac{\log p}{p^s}, \quad \Re(s) > 1 \]

PNT: \( \lim_{x \to \infty} \pi(x) \log x = 1 \)

Theorem 1. \( \lim_{x \to \infty} \frac{\mu(x)}{x} = 1 \iff \lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1 \)

Theorem 2. If \( \int_0^\infty (\frac{\mu(x)}{x} - 1) \frac{dx}{x} \) converges

then \( \lim_{x \to \infty} \frac{\mu(x)}{x} = 1 \).

Remark: If \( \int_0^\infty [\mu(e^t)e^{-t} - 1] \ dt \) converges

then \( \int_0^\infty (\frac{\mu(x)}{x} - 1) \frac{dx}{x} \) converges.

Let \( h(x) = \mu(e^t)e^{-t} - 1 \).
Then \( Lh(s) = \int_0^\infty \hat{h}(t) e^{-st} \, dt \), for \( \text{Re}(s) > 0 \)

\[
(1) \quad = \frac{\Phi(s+1)}{(s+1)} - \frac{1}{s}.
\]

\[
(2) \quad -\frac{L'(s)}{L(s)} = \Phi(s) + \sum_{\rho \in \rho} \frac{2\log(p)}{p^s(p^s-1)}.
\]

**Theorem 3** From (2) \( \Phi(s) \) extends meromorphically across \( \text{Re}(s) = 1 \) and hence \( Lh(s) \) extends meromorphically across \( \text{Re}(s) = 0 \).

The only possible poles of \( Lh(s) \) on \( \text{Re}(s) = 0 \) are at zeros of \( \Phi(s) \) on \( \text{Re}(s) = 0 \).

**Theorem 4** \( \Phi(s) \) has no zeros on \( \text{Re}(s) = 1 \).

**Corollary** \( Lh(s) \) extends analytically across \( \text{Re}(s) = 0 \).

**Theorem 5** (Tauberian Theorem) \( Lh(s) \) extends analytically across \( \text{Re}(s) = 0 \) implies that

\[
\lim_{t \to 0^+} \int_0^\infty \hat{h}(t) e^{-st} \, dt = Lh(0+).
\]

and hence converges.

By the remark, Theorem 2 and Theorem 1, PNT is true.