Zeroes of the Partial Sums of the Exponential Function

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Contents

1 Introduction

In this paper we present a summary of some surprising results about the zeroes of partial sums of the exponential function. We define the function

$$s_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$$

to be the n^{th} partial sum of the exponential function on the complex plane. The Fundamental Theorem of Algebra shows that each s_n has n zeroes counting multiplicities, whereas the actual exponential function has no zeroes on the entire complex plane. The paper falls into two parts. In the first part, we compute directly the behavior of the zeroes of these partial sums. Since most of the results from the first half are relatively unknown, we will go through the proofs in detail. In the second half of the paper, we show how this behavior in some sense uniquely characterizes the exponential function. Naturally this requires considerably more sophistication, and in proving the certain statements, we will need to introduce several large-caliber results including the Weierstrass and Hadamard Factorization Theorems. All figures were produced using Mathematica.

2 Asymptotic Behavior of Zeroes and the Zero Free Region

It becomes convenient to make the substitution

$$p_n(z) = s_n(nz).$$

We show in this section that the zeroes of p_n asymptotically fall along a horseshoe shaped curve in the unit circle and we obtain estimates on the rates of convergence. In this section we present a description of how the zeroes behave asymptotically. Specifically, we show that the zeroes of $p_n(z)$ (the scaled partial sums) asymptotically fall near the curve

$$\Gamma = \{z : |ze^{1-z}| = 1, |z| \leq 1\}.$$

A plot of Γ is shown in Figure 1 below.

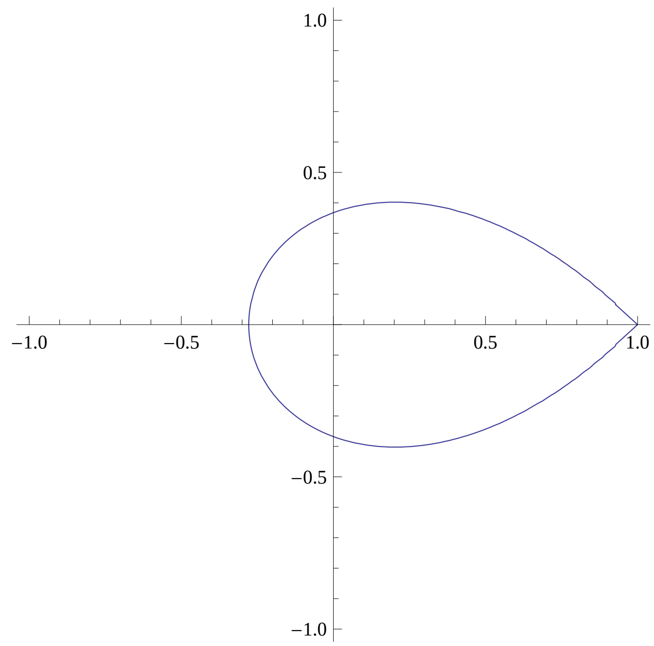


Figure 1: The cureve Γ.

Further we show that the zeroes always fall outside of the region bounded by Γ, and we obtain estimates on how quickly the zeroes approach the curve.

2.1 Lower Bound on Distance to Γ

The following lemma and theorem are modified from [5, Theorem 1], which is a refinement of an original result by J.D. Buckholtz in [1].

Lemma 2.1. *For all n, $p_n(1) > e^n/2$.*

Proof. The proof is mainly a little bit of cleverness in evaluating power series. Observe that

$$s_n(x)e^{-x} = \frac{1}{n!}\int_0^x t^n e^{-t}\,dt = 1 - \frac{1}{n!}\int_x^\infty t^n e^{-t}\,dt,$$

which can be verified simply by evaluating the integrals using integration by parts. Since we will be setting $x = n$, both integrals in the above expression will be positive, and hence it is sufficient to show that

$$\int_0^n t^n e^{-t}\,dtx < \int_n^\infty t^n e^{-t}\,dt. \tag{1}$$

Through a change of variables, we see this is equivalent to

$$\int_0^1 (te^{-t})^n\,dt < \int_1^\infty (te^{-t})^n\,dt.$$

By examining Taylor coefficients and performing some algebraic manipulation, we see that

$$e^u(1-u) < e^{-u}(1+u),$$

for all u such that $0 < u < 1$. Hence if $g(t) = te^{-t}$ then we have

$$g(1-u) < g(1+u)$$

which implies that

$$\int_0^1 (te^{-t})^n \, dt < \int_1^2 (te^{-t})^n \, dt,$$

which clearly implies (1). □

Theorem 2.2. *There are no zeroes z of p_n with $|z| \leq 1$ and $|ze^{1-z}| \leq 2^{1/n}$.*

Proof. Observe that it is sufficient to show that if $|z| \leq 1$ and $|ze^{1-z}| \leq 2^{1/n}$, then $|1 - e^{-nz}p_n(z)| < 1$, which implies that $p_n(z) \neq 0$. Compute as follows:

$$\begin{aligned} |1 - e^{-nz}p_n(z)| &= \left| e^{-nz} \sum_{k=n+1}^{\infty} \frac{(nz)^k}{k!} \right| \\ &= \left| \left(ze^{1-z}\right)^n e^{-n} \sum_{k=n+1}^{\infty} \frac{n^k z^{k-n}}{k!} \right| \\ &\leq 2e^{-n} \sum_{k=n+1}^{\infty} \frac{n^k}{k!} < 1 \end{aligned}$$

by the lemma, which thus implies that $p_n \neq 0$.

□

Corollary 2.3. *There is a number C such that s_n has no zeroes in the circle of radius Cn.*

Theorem 2.4. *There are no zeros z of p_n for which* $\operatorname{dist}(z, \Gamma) \leq (2^{1/n} - 1)/(2e^2)$ *and $|z| \leq 1$.*

Proof. Let $f(z) = ze^{1-z}$. Hence $f'(z) = (1-z)e^{1-z}$. Then if $\operatorname{dist}(z, \Gamma) \leq d = (2^{1/n} - 1)/(2e^2)$ and u is a point on Γ such that $|z - u| \leq d$, then we have

$$f(z) \leq |f(u)| + 2e^2 d = 2^{1/n}$$

so that applying the result from the previous theorem shows that there are no zeroes within $2^{1/n}$ of Γ which lie inside the unit disk. □

This theorem sets a bound on how quickly the zeroes of p_n can approach the curve Γ. At this point, we still haven't shown that the zeroes of s_n even approach Γ, but we have shown that the region bounded by Γ is free of zeroes and that the zeroes approach no faster than $2^{1/n}$.

2.2 Upper Bound on Distance to Γ.

The next natural question to ask is how quickly the zeroes of p_n approach Γ. We now present a several lemmas leading to a theorem due to Buckholtz in [1] and [2]. Let T_n be defined as

$$T_n(z) = \frac{n!}{(nz)^n} p_n(z).$$

Lemma 2.5. *The following equation is true for all $z \neq 0, 1$.*

$$T_n(z) = \frac{z}{z-1}\left(1 + \frac{T_n'(z)}{n}\right), \qquad z \neq 0, 1. \tag{2}$$

Proof. It is sufficient to show that

$$(z-1)T_n(z) = z\left(1 + \frac{T_n'(z)}{n}\right).$$

Computing the left side of this equation yields

$$\begin{aligned}(z-1)T_n(z) &= zT_n(z) - T_n(z)\\ &= \sum_{k=0}^{n} \frac{n^{k-n}n!}{k!} z^{k-n+1} - \sum_{k=0}^{n} \frac{n^{k-n}n!}{k!} z^{k-n}\\ &= z - \sum_{k=0}^{\infty}(k-n)\frac{n^{k-n-1}n!}{k!} z^{k-n}.\end{aligned}$$

Similarly, evaluating the right side of the equation yields that

$$\begin{aligned} z\left(1 + \frac{T_n'(z)}{n}\right) &= z\left(1 + \frac{1}{n}\left[\sum_{k=0}^{n} \frac{n^{k-n}n!}{k!} z^{k-n}\right]'\right)\\ &= z + \sum_{k=0}^{n} \frac{n^{k-n-1}n!}{k!} z^{k-n},\end{aligned}$$

which is clearly equal to what we get on the right hand side. Therefore (2) holds. ∎

Lemma 2.6. *If $|ze^{1-z}| \geq 1$, then*

$$|T_n(z)| < 2e\sqrt{n}.$$

Proof. The proof is relatively straightforward, but involves defining the function S_n as

$$S_n(z) = \frac{n!}{(nz)^n} \sum_{k=n+1}^{\infty} \frac{(nz)^k}{k!}.$$

A straightforward manipulation yields that yields that

$$T_n(z) + S_n(z) = \frac{n!(e/n)^n}{(ze^{1-z})^n}. \tag{3}$$

Applying the triangle inequality to the functions T_n and S_n yields that for $|z| \leq 1$, we have

$$|S_n(z)| \leq S_n(1)$$

and similarly for $|z| \geq 1$, we have

$$|T_n(z)| \leq T_n(1).$$

Further, since $T_n(1)$ and $S_n(1)$ are both positive both must be smaller than

$$n!(e/n)^n \leq e\sqrt{n},$$

where the latter inequality follows from Sterling's inequality. Applying the triangle inequality to the cases $|z| \leq 1$ and $|z| \geq 1$ separately, we see if $|ze^{1-z}| \geq 1$, then both of T_n and S_n are smaller in magnitude than $2e\sqrt{n}$.

□

Theorem 2.7. *For every integer n, the zeroes of of p_n lie within distance $2e/\sqrt{n}$ of Γ.*

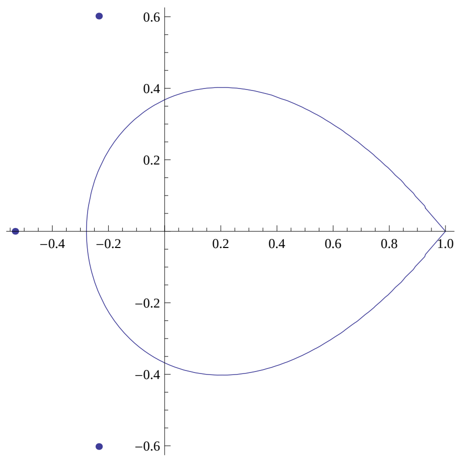
Proof. Notice that the zeroes of p_n are the same as those for T_n.

Since T_n is analytic outside of Γ and the region bounded by Γ is convex, we can use the Cauchy's Inequality for derivatives with the above lemma to see that if z is δ or farther away from Γ (and outside of the region bounded by Γ), then we have

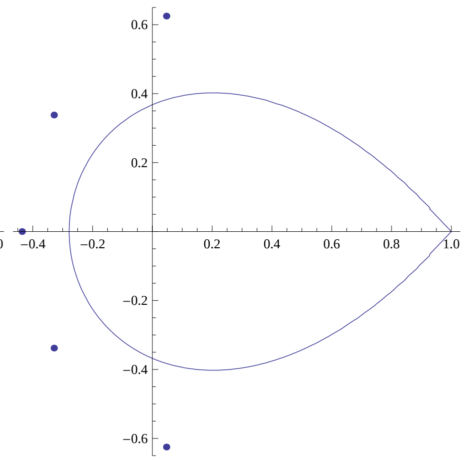
$$|T_n'(z)| \leq 2e\sqrt{n}/\delta.$$

Inserting this into (2) yields if $\delta > 2e/\sqrt{n}$, we have no zeroes for T_n and thus none for p_n. □

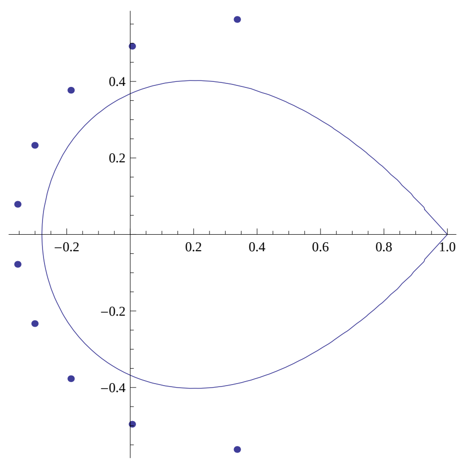
Together, the preceding theorems give us a good understanding of the behavior of the zeros of p_n. They show that the zeroes asymptotically and uniformly approach Γ and give us estimates on the rates of convergence. Below we show several renderings of the behavior of the zeroes of p_n for various values of n.



(a) Zeroes for $n = 3$, shown next to Γ.



(b) Zeroes for $n = 5$, shown next to Γ.



(c) Zeroes for $n = 3$, shown next to Γ.

Figure 2: Zeroes of p_n for $n = 3, 5, 10$.

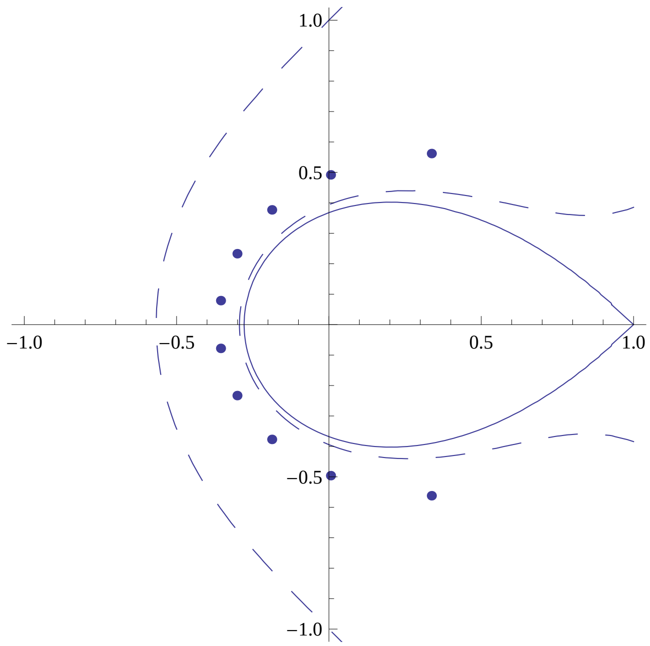


Figure 3: The $n = 10$ case with the proven outer and inner boundaries for the zeroes shown. The boundaries are rendered as dashed lines. Γ is shown as a solid line.

3 Characterization of the Exponential

A surprising feature of the exponential is that just a small amount of information on the behavior of the zeroes uniquely determines the exponential function. It is not surprising that knowing that the zeroes of p_n asymptotically fall along Γ would uniquely determine the exponential function, but perhaps more striking is that we can extend Corollary 2.3 to be an if and only if statement. In the following subsection, we present a discussion leading to the statements of the Weierstrass and Hadamard factorization theorems. We will discuss these theorems and give proofs where simple, but more involved proofs of more well known results are not of particular interest for this paper and thus can be found in [3] by the interested reader. These two theorems will lead to a result proved by Buckholtz in [1].

3.1 Factoring an Entire Function

For our discussion of the zeroes of the partial sums of the exponential function, we wish to be able to think of factoring an entire function which may have an infinite number of roots. The simplest of all cases is that of an entire function with no zeroes.

Theorem 3.1. *Let $f(z)$ be an entire function with no zeroes. Then $f(z) = e^{g(z)}$ for some entire function g .*

Proof. Let
$$g(z) = \int_0^z \frac{f'(\zeta)}{f(\zeta)}\, d\zeta + \mathrm{Log} f(0),$$
which we can observe the be analytic since the $f(z)$ is never zero, so the integrand is analytic. It is easy to verify that $e^{g(z)}$ and $f(z)$ both solve the differential equation
$$h'(z) = h(z)\frac{f'(z)}{f(z)}$$
with $h(0) = f(0)$. Therefore by the uniqueness principle for differential equations, f and e^g must be the same function. □

To consider functions that actually have zeroes, we need to introduce some machinery. Let

$$E_0(z) = (1 - z)$$
$$E_p(z) = (1 - z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right), p \geq 1.$$

Theorem 3.2. *If $\{a_n\}$ is a sequence in $\mathbb{C}$ such that $\lim |a_n| = \infty$ and $a_n \neq 0$ for $n \geq 1$. If $\{p_n\}$ is any sequence of integers such that*
$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty$$
for all $r > 0$, then
$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$
converges and is analytic. Furthermore, the sequence $p_n = n - 1$ will always yield convergence.

We usually call
$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$
the **canonical product**. The proof of this theorem is not particularly difficult, and relies mostly on relatively simple estimates on the convergence of the product, but is somewhat involved and will be omitted. The interested reader can find it in [3] or many other elementary texts on complex analysis.

Applying the previous theorems yields the following formulation of the Weierstrass Factorization Theorem.

Theorem 3.3 (The Weierstrass Factorization Theorem)**.** *Let f be an entire function and let $\{a_n\}$ be the zeroes of f repeated with appropriate multiplicity and suppose f has a zero of order m at $z = 0$. Then there is a sequence of integers p_n and an entire function g such that*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right). \tag{4}$$

Proof. The proof follows almost immediately from the previous theorems. From Theorem 3.2 we know there's a function h in the following form

$$h(z) = z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right),$$

which has the same zeroes as f with the same multiplicities. Therefore f/h is entire and nonzero, so by Theorem 3.1, we know that

$$\frac{f(z)}{g(z)} = e^{g(z)}$$

for some entire function $g(z)$, as we wanted to show. □

We now present several definitions and basic theorems leading to a statement of the Hadamard Factorization Theorem.

Definition. Let f be an entire function with zeroes $\{a_1, a_2, \dots\}$ repeated according to multiplicity and arranged such that $|a_1| \leq |a_2| \leq \cdots$. Then we say f is of **finite rank** if

$$\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$$

for some integer p. We say f is of **rank** p if p is the smallest integer which yields convergence in the preceding series.

Observe that if f is of finite rank then

$$P(z) = \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$$

will work as an expression for the canonical product in (4). Notice also that if we use the above expression for $P(z)$, then our expression for f is unique except for integer multiples of $2\pi i$ in $g(z)$. We say such an expression for $P(z)$ is in **standard form**.

Definition. An entire function f is said to have **finite genus** if f has finite rank and if

$$f(z) = z^m e^{g(z)} P(z)$$

where $P(z)$ is in standard form and g is a polynomial. If p is the rank of f and q is the degree of g, then we call $\mu = \max\{p, q\}$ the **genus** of f.

Definition. We say a function is of **finite order** if there is a constant $\alpha \in \mathbb{R}$ such that

$$|f(z)| \leq e^{|z|^\lambda}$$

whenever $|z|$ is sufficiently large. We call the infimum of all λ such that this is true the **order** of f.

We now cite an important theorem relating the genus of a function to it's order, as is stated in [3].

Theorem 3.4. *If f is an entire function of finite genus μ, then f is of finite order and $\lambda \leq \mu + 1$.*

The proof of this theorem involves several pages of computation, and thus will be omitted. It should be noted though, that despite it's length, the proof is relatively elementary and involves only relatively straightforward estimation. The interested reader can find it in [3].

We are finally ready to present the Hadamard Factorization as a converse to the previous result. We need this factorization theorem to prove a characterization of the exponential function through the roots of the partial sums.

Theorem 3.5. *Hadamard Factorization Theorem If f is an entire function of finite order λ then f has finite genus $\mu \leq \lambda$.*

Again, the proof of this theorem is relatively straightforward, but beyond the interest of this paper. The interested reader can find it in [3].

3.2 Characterizing the Exponential Function

In this section we prove a theorem about how the growth of the zeroes of the partial sums of the exponential function completely characterize the exponential function. First we need to state a lemma on calculating the order of a function.

Lemma 3.6. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then the order of f can be calculated by*

$$\lambda = \limsup_{n\to\infty} \frac{n \log n}{\ln(1/|a_n|)}.$$

We won't prove this lemma since the result is fairly well known. The interested reader can find a proof in [4] or the outline of the proof in an exercise in [3]. The following theorem represents the culmination of this entire paper. It originally was due to Buckholtz and can be found in [1].

Theorem 3.7. *Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an entire function. The following two statements are equivalent:*

(i) There is a positive number c such that for each n, the function $\sum_{k=0}^{n} a_k z^k$ has no zeroes with norm less than cn.

(ii) The function f can be represented as ae^{bz}.

Proof. We have already proved that (ii) implies (i), so we must just show the other direction. Let f be an entire function such that

$$f(z) = \sum_{k=0}^{\infty} a_n z^n$$

and let

$$s_n(z) = \sum_{k=0}^{n} a_k z^k.$$

Now suppose that there is a constant c such that s_n has no zeroes in the disk $|z| < cn$. By Hurwitz's theorem, f has no zeroes. Since $|a_n/a_0|^{1/n}$ is the geometric mean of the modulii of the zeroes of s_n, and since all of the zeroes are outside of the circle of radius cn, we know that

$$\left|\frac{a_n}{a_0}\right|^{1/n} > cn.$$

Taking logarithms and rearranging yields

$$\frac{\log n + \log c}{\frac{1}{n}\log|a_n| - \frac{1}{n}\log|a_0|}.$$

Separating out terms that go to zero yields that

$$\limsup_{n\to\infty} \frac{n\log n}{\log(1/|a_n|)} \leq 1.$$

Therefore, by the previous lemma, f has at most order 1. Theorem 3.1 shows us that

$$f(z) = e^{g(z)}$$

for some entire function g. By the Hadamard Factorization Theorem, g must be of degree less than or equal to 1. Hence $f(z) = ae^{bz}$ for some $a, b \in \mathbb{C}$, as we wanted to show. □

4 Conclusion

Hopefully at this point the reader understands how all of this material unifies itself. Though we had to make a diversion to study the Hadamard and Weierstrass factorization theorems, they lead us to Buckholtz's astonishing equivalence result concerning the exponential function. In particular, this shows that the zeroes of the partial sums of the exponential function increase faster than those of any other entire function.

References

[1] J. D. Buckholtz, *A Characterization of the Exponential Series*, The American Mathematical Monthly **73** (Apr., 1966), 121-123.

[2] ———, *Concerning an Approximation of Copson*, Proceedings of the American Mathematical Society **14** (Aug., 1963), 564-568.

[3] John B. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York, 1973.

[4] B. Ja. Levin, *Distribution of Zeros of Entire Functions*, American Mathematical Society, United States, 1980.

[5] Peter Walker, *The Zeros of the Partial Sums of the Exponential Series*, The American Mathematical Monthly **110** (Apr., 2003), 337-339.